ELIAKIM HASTINGS MOORE

An Appreciation by H. E. Slaught

In the death of Professor Eliakim Hastings Moore on December 30, 1932, the University of Chicago lost one of its most inspiring leaders and the world-wide fraternity of mathematicians lost one of its most gifted members. He was one of that remarkable company of scholars who from the very outset in 1892 determined the high character of the University of Chicago and gave to it great distinction. Professor Moore was one of the youngest of this group of men but his dauntless spirit, his boundless enthusiasm, his scientific integrity, and his unsurpassed logical insight soon established an influence which has extended far and wide and will continue to be felt wherever students of mathematics are seeking to discover the mainspring of their inspiration.

Looking back over a period of forty-one years of intimate association with Professor Moore, this writer, whose acquaintance with him began in the capacity of first fellow in the department of mathematics at the University of Chicago, now wishes to enumerate certain outstanding characteristics of this truly great man and, if possible, to set them forth with sufficient emphasis to command the attention and attract the emulation of the on-coming generation of students of mathematics.

First, and perhaps most important, was Professor Moore's exemplification of *intellectual honesty*. It was impossible for him to accept as established any proposition whose proof was in doubt or not fully shown, much less to promulgate any such proposition. There was no place for camouflage of any sort in his mental make-up. A striking and critical test of this characteristic occurred at the opening meeting of the Mathematical Club in one of the early summer quarters. Moore was presenting a paper on a highly technical topic to a large gathering of faculty and graduate students from all parts of the country. When half way through he discovered what seemed to be an error (though probably no one else in the room observed it). He stopped and re-examined the doubtful step for several minutes and then, convinced of the error, he abruptly dismissed the meeting—to the astonishment of most of the audience. It was an evidence of intellectual *courage* as well as *honesty* and doubtless won for him the supreme admiration of every person in the group—an admiration which was in no wise diminished, but rather increased, when at a later meeting he announced that after all he had been able to prove the step to be correct.

A second outstanding characteristic was the *intellectual freedom* which he claimed for himself and which he cheerfully granted to all others. He expected every man on the staff to do his full duty but he did not follow him up or visit his classroom. He trusted, rather, to the freedom based upon individual responsibility and, on the whole, the results were far more satisfactory than any system of espionage could possibly have been. There were many departmental open discussions where every instructor's views could be fully weighed and evaluated in the light of all other considerations; but in the end individual free-

dom was accorded to each one for working out the general plan. If in any case the results were unsatisfactory, it was up to the instructor to discover the trouble and to find the remedy. To this acid test of *freedom*, Professor Moore subjected himself on equal terms with all his instructors, even to the extreme test, on occasion, of admitting his own failure and seeking the remedy from the counsel of others.

Among many other characteristics which might be mentioned are: *fairness of judgment, unselfish devotion* to his students, and *whole-hearted confidence* in his chosen science. Professor Moore never passed judgment on any question until all the evidence had been examined and evaluated. If in any case it seemed that he had erred he was always ready to grant reconsideration in the light of new evidence. With his students, especially those who showed ability in research, his interest never flagged. He was constantly in touch with their progress and was a never failing source of inspiration. His favorite lecture period was eleven o'clock but he seldom heard or heeded the twelve o'clock bell, so engrossed was he in the development of the topic in hand, whether he himself or one of the class was making the report. His interest, however, was not limited to his own special students nor even solely to Chicago students or faculty. A common occurrence at the time of the mathematical meetings in Chicago was a scientific tête-à-tête in a secluded corner between Moore and some younger man who had presented a paper (perhaps his first attempt) showing worthy effort in independent thinking. His confidence in the worthwhileness of his chosen science was his supreme delight not only in an abstract, logical sense but also in its concrete significance for the comfort and progress of the human race, as he believed that it influences the thought and touches the daily life of mankind (though in most cases quite unconsciously) in a multitude of ways.

It is not the purpose of this writer to make any attempt at weighing or evaluating the scientific work of Professor Moore, nor even to give a bibliography of his published papers. These phases of his long life work will be treated by others in various publications. Suffice it to say here that he has left a great legacy of unpublished research material concerning General Analysis and that his published papers and abstracts include 29 in Group Theory, 10 in Geometry and Foundations, 17 in Function Theory, 11 in Integral Equations and General Analysis, and 7 on miscellaneous topics, making a total of 74. In this connection there should be emphasized Professor Moore's important and widely extended influence through his former students, and especially through the thirty-one doctors of philosophy whose theses were written under his supervision. Of these Ph.D.'s, there are three on the faculty at the University of Chicago, two each at the University of Iowa, and the University of the Philippines, one each at Harvard, Princeton, Cornell, Bryn Mawr, Vassar, Swarthmore, Toronto, Annapolis, Fisk University, the state colleges of South Dakota and Pennsylvania, and the state universities of Michigan, Wisconsin, Nebraska, Washington, Louisiana, Texas, Montana, and California. Three are deceased and two are not teaching, though they are engaged in research occupations. One of these

doctors, himself an internationally known mathematician, is quoted as saying: "For forty years Moore has been a leading figure in the mathematical world, not merely in fundamental research, but also in the nationalization of a local mathematical society, the institution and editing of its research journal, and finally inspiring and starting on their research careers so many of the younger generation. Moore's work easily places him among the world's greatest mathematicians. He was a leader who was universally loved because he was at the same time a prince of a man." Another of these doctors has said: "His genius for mathematical work, and the personal enthusiasm which spurred him on to great success, were inimitable and constituted a tremendous inspiration to all who came in contact with him, students and colleagues alike."

But there is another phase of Professor Moore's remarkable influence in the general domain of mathematics which might easily be overlooked, but which should not be underestimated, amidst the greater effulgence of his brilliant research career; namely, his deep interest in the teaching of mathematics in the secondary schools and colleges and his ardent support of the Mathematical Association of America and of this MONTHLY as its official journal dedicated to the interests of mathematics in the collegiate field. It will be worthwhile to trace the development of this pedagogical interest in Moore's mind. It began in 1894 when he gave personal encouragement to Benjamin F. Finkel who was then just starting the AMERICAN MATHEMATICAL MONTHLY. They both hoped that this new journal would attract and inspire high school teachers. In this they were greatly disappointed but the new periodical soon found a fitting place in the collegiate world. Moore and Dickson contributed articles in the early numbers which gave encouragement to Finkel and prestige to the MONTHLY and later, in 1902, with Moore's advice and consent, Dickson accepted the co-editorship and the University of Chicago began an annual subsidy contribution of $50.00 which was continued till the MONTHLY became self-supporting in 1915.

Meanwhile in 1894, the American Mathematical Society was organized in New York and in 1897 the first Section of this Society was formed at Chicago largely through Moore's initiative. In 1901–1902 Moore was president of the Society and in December, 1902, he delivered his presidential retiring address in which he revealed his profound interest in the pedagogical situation with respect to mathematics in America and set forth with his keen logic and amazing skill and insight the relationship that should subsist between so-called "pure" and "applied" mathematics, or in the words of Klein, between "mathematics of precision" and "mathematics of approximation." Here we find a mathematician whose keenest delight was in the realm of the most abstract logical analysis pleading with his scientific confrères in astonishing earnestness for the recognition and promulgation of the "laboratory method" in the teaching of mathematics in the elementary, secondary and collegiate-schools; that is, for the *concrete* approach to all *abstract* mathematical ideas. He firmly believed that the methods of presenting mathematics in the schools could thus be modified, greatly to their pedagogical advantage, and so as to win for this queen of

the sciences the "very high position in general esteem and appreciative interest which it assuredly deserves."

A re-reading of this presidential address[1] will convince the most casual observer that Professor Moore had given very earnest and deep thought to the question of improvement (yes, of reform) in the teaching of mathematics in the schools and that he had weighed carefully similar movements in Europe, especially in England and Germany. There is now good evidence that he was a seer and a prophet many years in advance of his time. While he recognized the chief responsibility of the American Mathematical Society to be the promotion of research, he nevertheless felt that even research interests in mathematics were bound up with those of mathematical education. He hoped that the Society might give more attention to the pedagogy of mathematics and he felt sure that such a movement would further the highest interests of mathematics in this country. But this movement was delayed for fourteen years and then was brought about by another procedure which will now be fitted into this picture.

It was shown above that Moore's interest in the founding of the MONTHLY was based on his desire for a general extension of mathematical education in this country. This interest again appeared when, in 1908, he urged this writer to undertake the co-editorship of the MONTHLY which Professor Dickson was then vacating to take up other editorial duties. Again in 1912 Moore's cooperation and council were of the greatest importance when the periodical was rescued from financial disaster and taken over by representatives of twelve universities and colleges in the middle west. His joy was like that of a child with a new toy when the first number, in January 1913, of the enlarged publication in its new dress was presented to him. And, finally, his interest and satisfaction were approaching an upper bound when, in January 1916, the MONTHLY became the official journal of the newly organized Mathematical Association of America. Ample evidence cropped up now and then to show that he was an habitual reader of the MONTHLY—problems and all. The first complete set of bound volumes of the periodical to be deposited in the Association library was the gift of Professor Moore and served as a token of his abiding interest in the movement for which the MONTHLY was to stand.

One of the first official acts of the Mathematical Association of America was the appointment of a committee under its auspices to consider the whole question of the teaching of mathematics in the secondary schools. It was composed of six representatives of universities or colleges and six from secondary schools. Professor Moore was an active member of this committee and his brother-in-law, J. W. Young, was its chairman. At last, after fourteen years of waiting, Moore found himself in the company of a nationally representative body wholeheartedly devoted to reform in the teaching of mathematics. This committee's activities were financed by the General Education Board and its findings were published in a voluminous report on the "Reorganization of Mathematics in

[1] *Bulletin of the American Mathematical Society*, Vol. 9, p. 402–424.

Secondary Education" whose circulation was nation-wide and whose influence soon became and still remains phenomenal though it is a decade since it was issued.

One outcome of this committee's influence was the organization of the National Council of Teachers of Mathematics whose membership now numbers nearly five thousand and whose activities have justified the responsibility which it inherited from the committee. Another outcome has been the firm establishment of the Mathematical Association of America as the effective intermediator between university research on the one hand and general mathematical education on the other hand. Professor Moore would have been the last to claim any special credit for these developments in the mathematical domain during the last thirty years, but it is nevertheless true that his personal influence has been potent and profound. The very fact that his life work was bound up in the highly technical and abstract reaches of his beloved science gave the world undeniable evidence that the interest which he manifested in the teaching of mathematics was genuine and whole-hearted.

Professor Moore was born at Marietta, Ohio, on January 26, 1862. He graduated at Yale University in 1883 and took his Ph.D. there in 1885. He spent a year in study at Berlin and Göttingen and taught for six years at Yale and Northwestern before coming to the University of Chicago as professor and acting head in 1892. He had been head of the department from 1896 to his retirement in 1931 as professor emeritus. His early associates at Chicago were Oskar Bolza and Heinrich Maschke. These three men supplemented each other perfectly. In the words of Professor Bliss: "Moore was brilliant and aggressive in his scholarship, Bolza rapid and thorough, and Maschke more deliberate but sagacious and without doubt one of the most delightful lecturers on geometry of all time."

Moore was the recipient of many honors, including an honorary Ph.D. from Göttingen in 1899, an LL.D. from Wisconsin in 1904, and honorary doctorates of science or mathematics from Yale, Clark, Toronto, Kansas and Northwestern. Besides his memberships in American, English, and German mathematical societies, he was a member of the American Academy of Arts and Sciences, the American Philosophical Society and the National Academy of Sciences. Two contributed funds have been established in his honor. The first is held by the American Mathematical Society for the purpose of assisting in the publication of his researches and for the establishment of a permanent memorial to him in the activities of the Society. The second fund has been expended for an unusually fine portrait of Professor Moore which hangs in Eckhart Hall at the University of Chicago. The interest in these funds displayed by his friends and admirers, in Chicago and throughout the country, was a remarkable tribute to him.

CURVES ARISING FROM A SINGLE INFINITY OF TRIANGLES

By MABEL M. YOUNG, Wellesley College

As basal figure, unchanged throughout the discussion, let us assume a parabola with those two of its tangents which pass through a given point in its plane. These two fixed tangents and a variable tangent of the parabola form a single infinity of triangles with which are associated many interesting curves intimately connected with the parabola. In the geometry of a triangle we are familiar with a vast number of points, lines and circles uniquely defined with respect to the three fundamental lines. Here we have a triangle in which two sides are held fast while the third varies continuously according to a simple law. We may ask what law of motion is thereby imposed on some of the significant points and lines associated with the triangle. To attempt a comprehensive study of this problem would indeed be to essay a formidable task. Almost any notable point or line, however, selected for investigation will prove rewarding and may be trusted to present most stimulating questions of theory for consideration. The writer has been interested chiefly in the projective properties of the figure. Free use has been made of theorems of elementary geometry which describe the special points and lines discussed but the curves arising from the motion of these points and lines have been derived by projective methods. They are novel only in that they are simple, familiar forms seen in unfamiliar relationships. The entire configuration has moreover a practical importance in that it offers an easily constructed group of projectively related forms which may to advantage be used for the discussion and illustration of a wide variety of geometric theorems.

Let the given parabola be denoted by π, the fixed tangents by p and q, and the variable tangent by t_i. The lines p and q meet in the fixed point T and t_i meets p and q in the two projective ranges of points, P_i and Q_i. The triangles pqt_i will be denoted by Δ_i, the orthocenters by H_i, the circumcenters by O_i, the centroids by M_i and the centers of the nine-point circles by N_i. The subscripts $i=1, 2, 3$—will be used to denote arbitrarily selected triangles with their associated points; $i=a, b, c$ will be used to designate triangles having special properties.

Known theorems relating to a parabola give useful preliminary information and need only to be stated. Since the altitudes of every triangle of tangents to a parabola meet on the directrix, the locus of the orthocenters of the triangles Δ_i is h, the directrix of π. Since the circle through the intersections of any three tangents of a parabola passes also through the focus, the circumcircles of Δ_i, which pass through T, pass also through a second fixed point, F, and form an elliptic system. Let this system of circumcircles be denoted by S_i. Conversely, any circle of S_i cuts p and q in points which determine a tangent of π. The locus of the circumcenters of Δ_i is o, the line of centers of S_i.

Associated with every triangle is a second, called its orthic triangle, which has as vertices the feet of the altitudes of the given triangle. We now consider the envelope of that side of the orthic triangles of Δ_i which lies opposite t_i. If in the triangles Δ_i we drop perpendiculars from the pairs of variable vertices, P_i and Q_i, upon the opposite sides we form two perspective pencils of parallel lines, corresponding rays of which meet on the directrix of π and determine on q and p two new projective ranges of points, Q_i' and P_i'. Since the ideal points of these ranges correspond the lines $P_i'Q_i'$ envelope a parabola. Thus *the envelope of that side of the orthic triangles of Δ_i which lies opposite t_i is a parabola tangent to p and q.* We shall denote this parabola by π', the tangents $P_i'Q_i'$ by t_i' and the triangles pqt_i' by Δ_i'.

The figure consisting of π and π' and the two series of tangent triangles, Δ_i and Δ_i', repays detailed examination. Associated with the triangles Δ_i we have the locus of orthocenters, which is the directrix of π, and the system of circumcircles through T and the focus. Similarly the locus of the orthocenters of Δ_i' is h', the directrix of π', and the system of circumcircles, S_i', passes through T and the focus, F'. From the geometry of a triangle it is evident that this system may also be described as consisting of the circles which have as diameters the segments from T to the orthocenters of the triangles Δ_i. Since the locus of these points is h, the directrix of π, we see that the circles S_i' pass through T and the foot of the perpendicular from T to that line. Thus F', the focus of π', lies on the directrix of π and the locus of the circumcenters of Δ_i' is a line, o', parallel to h and midway from T to h. The complete figure before us accordingly includes π, π' and their foci and directrices; Δ_i, Δ_i' and the two systems of circumcircles with the line of centers of each. We remark particularly the incidence of the foci and certain of these forms.

The interrelationships of this somewhat complicated configuration are clearly brought out by the use of a certain involution of rays on T. Since the circumcenter and orthocenter of any triangle are a pair of isogonal points as to the triangle the lines joining these points to a vertex make equal angles with the bisectors of the corresponding angle. Thus the lines joining T to the pairs of points O_i and H_i of the triangles Δ_i form an involution of which the bisectors of the fixed angle (pq) are the double rays. We shall denote this involution by I. We show first that the lines from T to the foci of π and π' are a pair in the involution. To that ray TO_i which passes through F must correspond a ray TH_i which is parallel to the axis of π. Since TF' is perpendicular to h, TF' is the correspondent of TF. As an incidental result of this fact we see that TF must be perpendicular to the directrix of π' and that the four lines o, h, o', h' are parallel in pairs, o to h', o' to h. Moreover since the triangles Δ_i and Δ_i' have the sides p and q in common and TO_i' falls on TH_i it follows that each line-pair in the involution TH_i, TO_i arising from the triangles Δ_i, coincides with the line-pair TO_i', TH_i' arising from the triangles Δ_i'. Hence the lines from T to the orthocenter and circumcenter of any triangle of Δ_i meet the lines o' and h' in the circumcenter and orthocenter of the corresponding triangle of Δ_i'.

The relationships just established make possible a very simple construction for π and π' by means of the two systems of circles, S_i and S_i'. We may assume as S_i a system of circles on any two points, T and F. Through T are drawn two lines, p and q. The circles S_i cut p and q in points which determine the tangents of a parabola, π. Since the vertex tangent of a parabola is the pedal of the focus as to the curve the circle with diameter TF determines this tangent. The directrix may then be drawn. A perpendicular from T upon h gives the second base point of the system S_i' by which the tangents of π' may be constructed. Any point, O_1, on the line of centers of S_i determines a circle S_1 which cuts p and q in the points where these lines meet t_1. The perpendicular from T upon t_1 cuts the line of centers of S_i' in O_1' and hence determines the circle S_1' which passes through the intersections of p and q with t_1'. To the circle with diameter TF, which determines the vertex tangent of π, corresponds the circle of S_i which has diameter TF' and determines the vertex tangent of π'. The circle through TFF' determines the third common tangent[1] of π and π'. Those circles of S_i which are tangent at T to p and to q meet q and p again in the points at which these lines touch π.

Projective relationships among the forms just described give rise to many curves projective with π. First of all it is clear that π is projectively related to π'. This follows at once from the fact that the two series of tangents t_i and t_i' determine on each of the common tangents p and q two superposed projective ranges. That is, $P_i \overline{\wedge} P_i'$, $Q_i \overline{\wedge} Q_i'$. To p and q considered as tangents of π correspond q and p considered as tangents of π'. To the third common tangent correspond two distinct lines. The ideal tangent is self-corresponding. Closely connected with the projectivity of π and π' is the projectivity of the systems of circles, S_i and S_i'. It is evident that two circles S_1 and S_1' having radii TO_1 and TO_1' on two corresponding rays of the involution I have tangents at T which are equally inclined to the bisectors of the angle (pq). The tangent of S_1 at T accordingly makes with the chord TF an angle equal to that between the tangent of S_1' at T and TF'. Thus the common chord of the system S_i subtends an angle in every circle S_1 equal to the angle subtended by the common chord of the system S_i' in the circle S_1'. Hence the systems are projectively related. We may note that the degenerate circles of the systems correspond. The finite and ideal chords of the one correspond respectively to the finite and ideal chords of the other. The projectivity of the conics π and π' and of the two systems of circles S_i and S_i' leads at once to the consideration of the curves arising from the intersection of corresponding elements. These will be discussed in a later section.

Projective ranges on the most important auxiliary lines of our figure give rise to numerous conics projective with π. Of great assistance in the determination of such conics are the many superposed projective flat pencils having T as base point. Such pencils are those formed by (a) the lines through T parallel to

[1] This easily constructed figure furnishes a convenient means of verifying the known properties of the pencil of parabolas touching three finite lines.

the bases t_i of the triangles Δ_i; (b) the perpendiculars from T upon t_i; (c) the medians from T to t_i; (d) the symmedians from T to t_i; (e) the radii of the circumcircles of Δ_i through T; (f) the tangents of the circumcircles at T. If in these pencils corresponding rays are defined as those arising from the same triangle of Δ_i, it is clear that any two of these six pencils are projectively related and that each is projective to the tangents of π and π'. The corresponding rays of certain pencils are also so related as to form involutions. Thus (a) and (b); (a) and (c); (c) and (d); (e) and (f) are pencils in involution. We notice among these the involution TO_i, TH_i of which frequent use has already been made.

We may at once employ this involution I to determine a particular group of conics projective with π. We have seen that every line-pair of I meets o, h, o', h' in four especially defined points O_i, H_i, O_i', H_i'. The four ranges traced by these points are clearly projective with one another and hence give rise to the four conics which are the envelopes of the lines O_iO_i', H_iH_i', O_iH_i, $O_i'H_i'$. The conics are parabolas and are tangent to the double lines of the involution I. Of these curves the most interesting is the envelope of O_iH_i. Since the line joining the circumcenter and orthocenter of a triangle is its Euler line, we see that this parabola is the envelope of the Euler lines of the triangles Δ_i. The Euler line of a triangle passes also through the centroid and nine-point center. The segment OH is bisected by N and divided by M in the ratio $-\frac{1}{2}$. The two point-pairs MH and ON divide each other harmonically. To find the significance of these relationships for the parabola enveloped by O_iH_i we consider those tangents of the curve which with o, h and the ideal tangent have an anharmonic ratio of -1 and $-\frac{1}{2}$ respectively. Specifically we determine n and m where $(oh, nl_\infty) = -1$ and $(oh, ml_\infty) = -\frac{1}{2}$. Then the segment of every tangent of the parabola intercepted between o and h is bisected by n and divided by m in the ratio $-\frac{1}{2}$. Thus n is the locus of the nine-point centers of Δ_i and m the locus of the centroids. Since $(MH, ON) = -1$, we see that m, h, o and n are four harmonic tangents of the parabola. Hence, *the Euler lines of the triangles Δ_i envelope a parabola of which the loci of the centroids, orthocenters, circumcenters and nine-point centers are four harmonic tangents.*

Another conic projective with π is enveloped by the lines joining the mid-points of the segments determined by p and q on a pair of corresponding tangents of π and π'. For each parabola the locus of such mid-points is a straight line tangent to the curve. We denote these loci by t_m and t_m'. If we joint to T the points in which t_i and t_i' meet t_m and t_m' respectively we have the pencils of medians of Δ_i and Δ_i'. Since the latter coincides with the pencil of symmedians of Δ_i, corresponding pairs of lines are equally inclined to the bisectors of angle (pq) and form an involution. Thus the mid-points of the segments on t_i and t_i' trace projective ranges on t_m and t_m'. It may be shown that in this projectivity the ideal points of the ranges correspond. The curve enveloped by the lines joining pairs of corresponding points is a parabola, tangent to t_m and t_m' and to the double lines of the involution. This result may be stated otherwise. From the geometry of a triangle it is known that the diameter of the nine-point circle

which passes through the mid-point of a base bisects the opposite base of the orthic triangle and the segment from the opposite vertex to the orthocenter. In our figure the locus of the latter point is o' and the locus of N is n. Thus the lines of our envelope, which pass through the mid-points of the segments on t_i and t_i', pass also through N_i' and O_i'. The ranges determined on t_m and o' are projective since corresponding points are joined to T by corresponding rays of two projective pencils. The line o' is accordingly a tangent of the curve. Since n bisects the segment intercepted between t_m and o' on every tangent, n also touches the parabola. Hence, *that diameter of the nine-point circle of every triangle Δ_i which passes through the mid-point of the variable base envelopes a parabola tangent to the bisectors of the fixed angle (pq), to t_m, t_m', o' and n.*

Many curves covariant with the variable triangles Δ_i and of order higher than the second arise from the union of corresponding elements in pairs of projectively related forms of the figure. Some of these curves however break into curves of lower order because of self-corresponding elements in the projectivities. The two systems of circumcircles, S_i and S_i', are a special case of two projective systems of circles. Here the systems have a base point in common and the circles of infinite radius correspond in such a manner that the finite chord of one corresponds to that of the other and the ideal chords correspond. Thus instead of the bicircular quartic, which is in general the locus of the intersections of corresponding circles in such systems, we have a circular cubic with double point at T and the ideal line. Corresponding tangents of π and π' also meet in a degenerate curve of order four. Since through T and the ideal point of each bisector of angle (pq) pass two pairs of corresponding tangents of the parabolas, these points are double points on the new locus. Since the ideal tangents are self-corresponding, the curve breaks into this line and a rational cubic passing twice through T and once through each of these ideal points. Of the many rational cubics which arise from the intersection of a pencil of rays on T and the tangents of a projectively related conic, we may mention the pedal of T as to π and the strophoids which are the pedal curves of T as to the group of parabolas tangent to the bisectors of angle (pq). Such examples might be multiplied indefinitely.

We make use of the method just described to determine the locus of the symmedian points of the triangles Δ_i. The symmedian point of a triangle is defined as the isogonal conjugate of the centroid. It may also be constructed as the intersection of the lines joining the mid-point of a base to the mid-point of the corresponding altitude. One of several convenient ways of determining the desired locus is to consider the symmedian point of a triangle of Δ_i as the intersection of the symmedian line through T and the line joining the mid-point of the base TP_i to the mid-point of the altitude Q_iP_i'. Since the locus of the mid-points of the altitudes Q_iP_i' is a line through T, the successive pairs of mid-points trace on this line and on p two ranges projective to each other, to P_i and to Q_i. Thus the lines joining the mid-points of the bases on p to the mid-points of the corresponding altitudes on p envelope a conic projective to π and

to the pencil of symmedians on T. Hence, *the locus of the symmedian points of the triangles Δ_i is a rational cubic with double point at T.* Since in the triangles Δ_i the medians and corresponding symmedians on T form a pencil in involution, we see that *the line which is the locus of the centroids of the triangles Δ_i and the rational cubic which is the locus of the symmedian points are projectively related forms of which the centroid and symmedian point of each triangle are a pair of corresponding elements.*

The special relationships which may exist among the forms we have derived are illustrated by the following theorems. The first is based on the conclusions of the preceding paragraph. Since a conic is determined by three tangents and a focus and the second focus is the isogonal conjugate of the first as to the three lines we see that *tangent to the three sides of each triangle of Δ_i is a unique ellipse having the centroid and symmedian point of the triangle as foci. The principal axes of these ellipses envelope a curve of fourth class.* The second theorem is an application of the familiar fact that the parabola determined by four lines has as focus the point of intersection of the circles circumscribing the four triangles formed by the lines and as directrix the line on which lie the orthocenters of these triangles. We consider any triangle Δ_i with the three altitudes P_iQ_i', Q_iP_i', TT_i'. A parabola touching p, q, P_iQ_i', Q_iP_i' has as focus T_i' and as directrix $P_i'Q_i'$. But T_i' is a point on the pedal of T as to π; $P_i'Q_i'$ is a tangent of π', and there is projective correspondence between the pairs T_i' and t_i'. Hence, *the single infinity of parabolas which touch the fixed sides of Δ_i and the successive pairs of altitudes upon the fixed sides have as corresponding foci and directrices pairs of projectively correlated elements on the pedal of T as to π and on π'.* Similarly we may show that *the single infinity of parabolas which touch the common tangents of π and π', p and q, and the successive pairs of tangents t_i and t_i' have as corresponding foci and directrices projectively correlated elements on the cubic derived from the systems S_i and S_i' and on the parabola enveloped by the line H_iH_i'.*

In conclusion one example may be given of the infinite variety of curves to be derived from π and the triangles Δ_i by the use of different methods. Let us take the polar reciprocal of π as to the polar circle of a triangle Δ_1. This circle has its center at the orthocenter of the triangle and with respect to it the triangle is self-conjugate. The parabola is reciprocated into a rectangular hyperbola having asymptotes parallel to the rectangular tangents drawn from H_1 to π. It passes through the vertices and orthocenter of Δ_1 and is tangent at H_1 to the directrix of π. Since the inverse of any point of the circumcircle of a triangle as to the polar circle is the corresponding point on the nine-point circle, then the polar of F, a point on the circumcircle of Δ_1, is the line perpendicular to FH_1 at its mid-point, I_1, through which pass the nine-point circle of Δ_1 and also the vertex tangent of π. But since I_1 falls on v, a perpendicular to FI_1 at I_1 is itself a tangent of π. Hence F is that point of the rectangular hyperbola which corresponds to this tangent of π. When four points of a rectangular hyperbola lie on a circle the point diametrically opposite any one is the orthocenter of the

other three. The points F and H_1 are accordingly ends of a diameter of the hyperbola, I_1 is its center and the tangents at F and H_1 are parallel. Thus the polar reciprocals of π as to the polar circles of the successive triangles Δ_i form a single infinity of rectangular hyperbolas passing through the fixed points T and F, tangent at F to the latus rectum of π and having the directrix of π as a second common tangent. The locus of the centers is the vertex tangent of π. The asymptotes of any curve of the system are parallel to the two perpendicular tangents which may be drawn to π from the point in which the hyperbola touches the directrix of π. We wish to show that these asymptotes have an envelope. Let us consider the pair of perpendicular tangents t_1 and t_2 drawn to π from a point, H_1, on h. These lines meet v in two points, V_1 and V_2, such that FV_1, FV_2 are at right angles to t_1 and t_2. The asymptotes of the corresponding hyperbola, center I_1, are also perpendicular to FV_1, FV_2 and pass through the mid-points of these segments. Thus as the orthocenters of Δ_i and the centers of the corresponding hyperbolas move on the directrix and vertex tangent of π, the feet of the perpendiculars from F on the asymptotes of the successive hyperbolas fall on a line, g, parallel to v and halfway from F to v. If then the various points of v are joined to F the perpendicular to each segment at its mid-point is an asymptote of some hyperbola of the system considered. These asymptotes accordingly envelope a parabola having F as focus, g as vertex tangent and the vertex tangent of π as directrix. The parabola is similar to π and similarly placed, ratio $\frac{1}{2}$.

THERMOSTATIC CONTROL

By J. L. SYNGE, University of Toronto

1. *Introduction.*

The function of thermostatic control of a heating system is to maintain a constant temperature, and, when the temperature is not at the desired level, to bring the temperature to the desired level and to keep it there. We treat here the case of a room, heated by radiators which are heated by an oil-burner. With an ideal thermostatic control, the burner would operate whenever the temperature of the room fell below the desired level, and would cease to operate when that level was exceeded. Actually, two controlling temperatures (or thermostatic temperatures) are involved, namely, a temperature α_1 at which the burner commences to operate when the temperature is falling and a temperature α_2 at which it ceases to operate when the temperature is rising; and α_2 is greater than α_1, although the difference is small. It is obvious from a practical point of view that if the oscillations of the temperature of the room are large, we are justified in neglecting $\alpha_2-\alpha_1$, and treating the problem as one in which an ideal control with a single temperature α is involved. But when the oscillations are small this is no longer the case. However the problem furnished by

the ideal control is of interest for its own sake, and we shall consider it as well as the case of more practical interest where $\alpha_2-\alpha_1$ is not zero.

For the ideal control ($\alpha_1=\alpha_2$) it is found that if the temperature of the external atmosphere is constant, the temperature of the room oscillates about the desired level, the amplitude of the oscillations and their periodic time both tending to zero as the time increases indefinitely.

For the practical control ($\alpha_1\neq\alpha_2$) it is found that if the temperature of the external atmosphere is constant, the oscillations of the temperature of the room tend asymptotically to a state of steady oscillation, whose amplitude is of the order of $h^{2/3}$ and whose periodic time is of the same order, where $h=\alpha_2-\alpha_1$, assumed small. The limiting amplitude and periodic time are evaluated in terms of the constants of the system. There is an interesting and simple relation between the intervals during which the burner is, and is not, in operation in the steady oscillation: we find in fact that the ratio of these intervals is equal to the ratio of the differences between the temperature for which the thermostat is set and two fundamental temperatures. These fundamental temperatures are (i) the temperature of the external atmosphere, (ii) the temperature to which the room would tend were the burner to remain in continuous operation.

The problem of these thermal oscillations has an exact dynamical analogue in the oscillations of a damped harmonic oscillator of the dead-beat type, subject to a discontinuous disturbing force. Considering for simplicity the case where the two thermostatic temperatures coincide (ideal control), this disturbing force is a repulsive force of constant intensity when the particle approaches nearer to the centre of oscillation than a certain position (corresponding to the thermostatic temperature) and is zero when the particle is on the other side of this position. The ultimate fate of the particle is to perform oscillations of infinitesimal amplitude and periodic time about what we may call its "thermostatic position." This position can hardly be called a position of equilibrium—we do not define the force acting on it in that precise position.

The dynamical analogue to the case where the thermostatic temperatures are different is easily described.

2. *The system.*

The thermal system consists of four parts:

(1) the burner;

(2) the water in the radiators (temperature $=\theta_2$);

(3) the atmosphere in the room (temperature $=\theta_3$);

(4) the external atmosphere (temperature $=\theta_4$).

We assume, for simplicity, a uniform temperature throughout the water and a uniform temperature throughout the room. If we assume that, when in operation, the burner supplies heat at a uniform rate to the water, and that the linear law of cooling is applicable throughout, the variations of temperature in the system are governed by the differential equations,

$$(1)\qquad \begin{cases} \dfrac{d\theta_2}{dt} = \epsilon k_{21} - k_{23}(\theta_2 - \theta_3), \\ \dfrac{d\theta_3}{dt} = k_{32}(\theta_2 - \theta_3) - k_{34}(\theta_3 - \theta_4), \end{cases}$$

where $\epsilon = 1$ when the burner is in operation and $\epsilon = 0$ when the burner is not in operation; k_{21}, k_{23}, k_{32}, k_{34} are *positive* constants of the system; θ_4 is supposed to be a known function of t.

If we denote by α_1, α_2, $(\alpha_2 > \alpha_1)$ the operating temperatures of the thermostat, then

$$(2)\qquad \epsilon = 1 \text{ if } \theta_3 < \alpha_1; \qquad \epsilon = 0 \text{ if } \theta_3 > \alpha_2.$$

In the range $\alpha_1 < \theta_3 < \alpha_2$, ϵ is determined through the previous history of the system by the condition that ϵ *always retains its value except when compelled to change it in order to satisfy* (2).

The solutions of the differential equations (1) are uniquely determined if the values of θ_2, θ_3 are assigned for $t = 0$, provided that θ_3 lies outside the range (α_1, α_2); if θ_3 lies inside this range, we require also to be told the initial value of ϵ.

3. *Reduction to dynamical form.*

The equations (1) may be replaced by the equivalent equations

$$(3)\qquad \frac{d^2\theta_3}{dt^2} + K_1\frac{d\theta_3}{dt} + K_2\theta_3 = \epsilon K_3 + f(t),$$

$$(4)\qquad \theta_2 = \frac{1}{k_{32}}\frac{d\theta_3}{dt} + \left(1 + \frac{k_{34}}{k_{32}}\right)\theta_3 - \frac{k_{34}}{k_{32}}\theta_4,$$

where

$$(5)\qquad \begin{cases} K_1 = k_{23} + k_{32} + k_{34} > 0, \\ K_2 = k_{23}k_{34} > 0, \\ K_3 = k_{21}(k_{32} + k_{34}) > 0, \\ f(t) = k_{34}\left(\dfrac{d\theta_4}{dt} + k_{23}\theta_4\right). \end{cases}$$

Equation (3) is the equation of motion of a damped harmonic oscillator, subject to the discontinuous disturbing force ϵK_3 and the varying disturbing force $f(t)$. Since temperatures must be continuous, the particle undergoes no abrupt change of position (θ_3), and, by (4), no abrupt change in velocity $(d\theta_3/dt)$, since θ_2 is to be continuous: that is to say, no impulsive forces are operative.

Let us assume that the temperature (θ_4) of the external atmosphere is constant. Let us put

$$\theta_4 = \frac{C}{K_2}, \quad f(t) = C. \tag{6}$$

The roots of the characteristic equation of (1), namely

$$\lambda_1 = \tfrac{1}{2}(-K_1 + \sqrt{K_1^2 - 4K_2}),\ \lambda_2 = \tfrac{1}{2}(-K_1 - \sqrt{K_1^2 - 4K_2}), \tag{7}$$

are seen, by (5), to be both real and negative. Thus when $\epsilon = 1$ or when $\epsilon = 0$, the oscillations given by (3) are of the dead-beat type—that is to say, there is at most one maximum or minimum of θ_3 as long as ϵ maintains the same value. The centres of the dead-beat oscillation are

$$\begin{cases} \theta_3 = \dfrac{K_3 + C}{K_2} \text{ when } \epsilon = 1; \\ \theta_3 = \dfrac{C}{K_2} \text{ when } \epsilon = 0. \end{cases} \tag{8}$$

These positions have simple interpretations, mechanically and thermally. Mechanically, they represent the equilibrium positions of the particle with and without the disturbing force K_3, provided that these positions lie in each case in the appropriate region. Thermally, the former represents the highest steady temperature $\bar{\theta}$ at which the burner, in continuous operation, can maintain the room, and the latter represents the steady temperature of the room when the burner does not operate at all—namely, the external temperature θ_4.

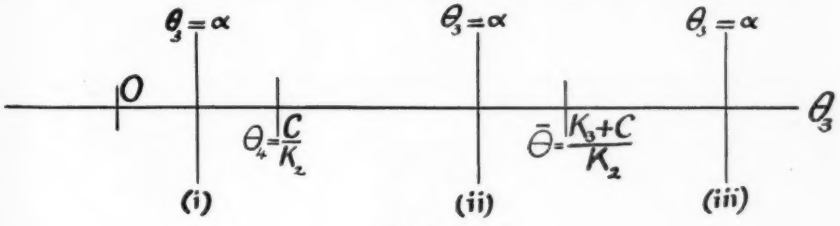


Fig. 1.

We may consider three cases, shown in Fig. 1., α_1 and α_2 being situated close to α:

(i) $\alpha_1 < \alpha_2 < \theta_4$: here the thermostat is set for a temperature below the external temperature.

(ii) $\theta_4 < \alpha_1 < \alpha_2 < \bar{\theta}$: here the thermostat is set for a temperature intermediate between the external temperature and the highest steady temperature to which the burner can heat the room.

(iii) $\bar{\theta}<\alpha_1<\alpha_2$: here the thermostat is set for a temperature above the highest steady temperature to which the burner can heat the room.

Cases (i) and (iii) are of little interest, practically or theoretically. In Case (i), it is easily seen that after at most one stationary value of θ_3, there is a dead-beat motion to θ_4, to which temperature the room tends steadily. Case (iii) is similarly discussed, and we may state the rather obvious conclusions:

In Case (i) the temperature of the room ultimately tends steadily to the external temperature θ_4.

In Case (iii) the temperature of the room ultimately tends steadily to the highest steady temperature $\bar{\theta}$ at which the burner can maintain it.

4. *The general nature of the oscillations.*

Let us turn to the interesting and practical Case (ii) where

$$\theta_4 < \alpha_1 < \alpha_2 < \bar{\theta}, \quad \theta_4 = C/K_2, \quad \bar{\theta} = (C + K_3)/K_2. \tag{9}$$

It is easy to see in a general way how the system behaves. Let us suppose, for definiteness, that initially $\theta_3<\alpha_1$. We start, then, with a dead-beat motion to $\theta_3=\bar{\theta}$. When $\theta_3=\alpha_2$ is encountered, this motion is replaced by a dead-beat motion to $\theta_3=\theta_4$. This carries θ_3 back across the position $\theta_3=\alpha_1$, and the original type of motion is resumed. It is clear that *the system oscillates across the thermostatic range $\alpha_1<\theta_3<\alpha_2$, and there is an infinite number of oscillations.* If we plot θ_3 against t, the graph will be a sinuous continuous curve, with a continuous first derivative, crossing the strip $\alpha_1<\theta_3<\alpha_2$ an infinite number of times. The second derivative has a discontinuity every time the curve crosses $\theta_3=\alpha_1$ from above or $\theta_3=\alpha_2$ from below.

To study the oscillations, it is convenient to introduce a change of variable suggested from mechanics. Let us write

$$v = \frac{d\theta_3}{dt}, \tag{10}$$

so that

$$\frac{d^2\theta_3}{dt^2} = v\frac{dv}{d\theta_3}. \tag{11}$$

Our differential equation (3) transforms into

$$v\frac{dv}{d\theta_3} + K_1 v + K_2\theta_3 = \epsilon K_3 + C; \tag{12}$$

this has the integrals

$$\left[v - \lambda_1\left(\theta_3 - \frac{\epsilon K_3 + C}{K_2}\right)\right]^{\lambda_1} = A\left[v - \lambda_2\left(\theta_3 - \frac{\epsilon K_3 + C}{K_2}\right)\right]^{\lambda_2}, \tag{13}$$

where λ_1, λ_2 are given by (7) and A is an arbitrary constant.

Let us now consider a graphical representation of the history of the system on a plane in which θ_3, v are taken as rectangular coordinates. This is, in fact, the "q, p" representation of the dynamical problem. Each possible history of the system is represented by a curve, which is continuous (since θ_3 and v are continuous), but which has discontinuities in direction on crossing $\theta_3=\alpha_1$ with $v<0$, and on crossing $\theta_3=\alpha_2$ with $v>0$. The sense of description of a curve is given by (10), which shows that since $dt>0$, θ_3 increases or decreases according as v is positive or negative. Equation (13) (with the appropriate values of ϵ) gives a singly infinite family of curves, covering the plane; the strip $\alpha_1<\theta_3<\alpha_2$ is covered twice.

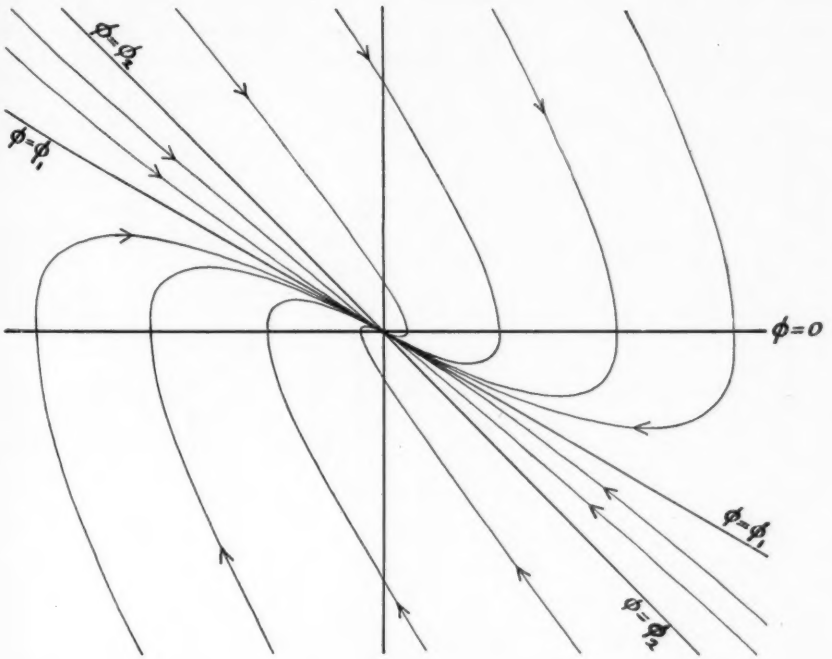


Fig. 2.

It is convenient to think of the complete families, given by substituting $\epsilon=1$ and $\epsilon=0$ in (13). The actual family would be given by drawing these complete families and then blotting out the part of the family $\epsilon=1$ for which $\theta_3>\alpha_2$, and the part of the family $\epsilon=0$ for which $\theta_3<\alpha_1$.

The family $\epsilon=1$ may be obtained from the family $\epsilon=0$ by a translation through a distance K_3/K_2 in the direction of the axis of θ_3. Either family may be obtained from a single member of it by a magnification with respect to the pole, which for $\epsilon=1$ is at $\theta_3=(K_3+C)/K_2$, $v=0$ and for $\epsilon=0$ at $\theta_3=C/K_2$, $v=0$. In

polar coordinates ρ, ϕ, with the pole as origin and the axis of θ_3 as initial line, the equation of either family is

$$\rho^{\lambda_1-\lambda_2} = B[\sin(\phi - \phi_2)]^{\lambda_2}[\sin(\phi - \phi_1)]^{-\lambda_1}, \tag{14}$$

where B is an arbitrary constant and

$$\tan\phi_1 = \lambda_1, \quad \tan\phi_2 = \lambda_2. \tag{15}$$

The lines $\phi=\phi_1$, $\phi=\phi_2$ (which lie in the second quadrant, since λ_1, λ_2 are negative) also belong to the family: they are touched by the other curves of the family at $\rho=0$ (for $\phi=\phi_1$) and at $\rho=\infty$ (for $\phi=\phi_2$). The curves meet the axis of θ_3 orthogonally, except that the pole, where they are tangent to $\phi=\phi_1$. The curves[1] are plotted in Fig. 2, the sense of description being shown by the arrows.

Fig. 3 shows the nature of the oscillations about $\theta_3=\alpha$ in the case of ideal control ($\alpha_1=\alpha_2=\alpha$).

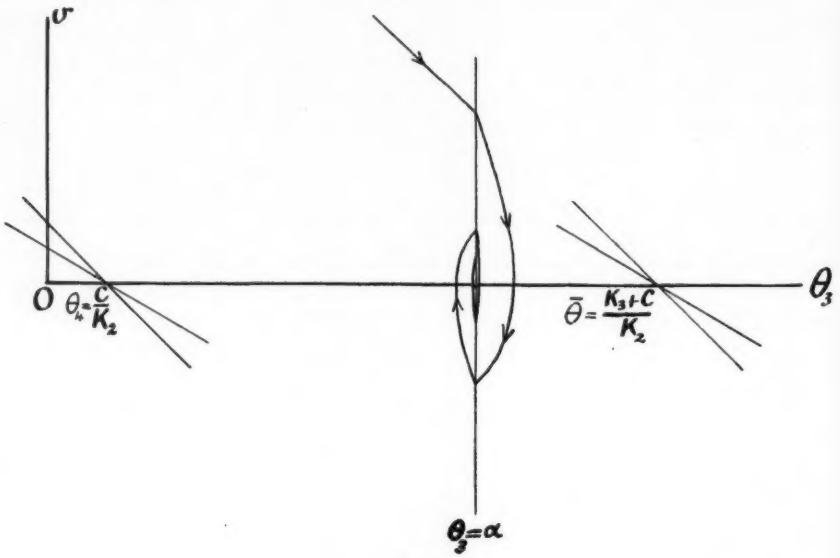


Fig. 3.

5. *The general nature of the ultimate oscillations.*

It is natural to expect that the temperature of the room tends, with indefinitely increasing time, to some definite temperature or to some simple type of oscillation. Indeed, we expect, in the case of ideal control, that the temperature tends to coincidence with the thermostatic temperature. In the case of the

[1] Although Figs. 2 and 3 are actual plottings, the values of the constants ($\phi_1=150°$, $\phi_2=135°$) were selected with a view to securing clear figures rather than a good approximation to conditions likely to arise in practice.

practical control, it is more difficult to see beforehand what the ultimate state will be.

Let us consider first the case of ideal control ($\alpha_1=\alpha_2=\alpha$). Let us suppose that the system passes from lower temperatures to the thermostatic temperature ($\theta_3=\alpha$) at the point A (Fig. 4), passes on to a maximum temperature (say

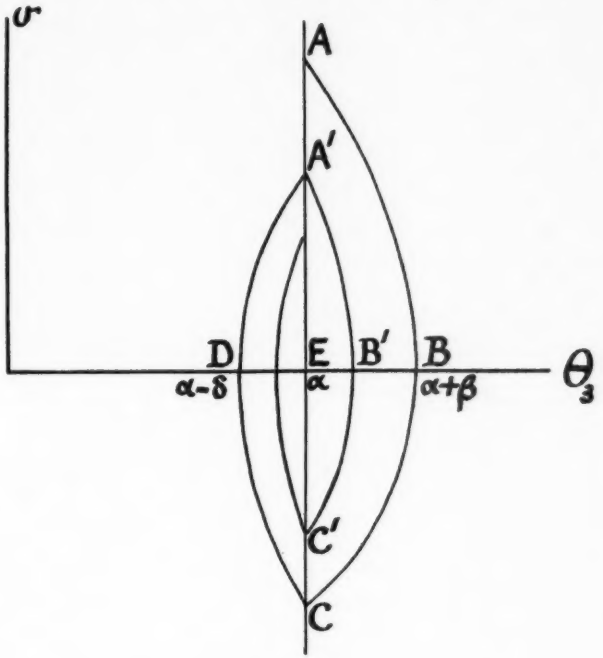


Fig. 4.

$\theta_3=\alpha+\beta$) at B, and falls again to the thermostatic temperature at C. The burner does not operate during this process, and hence the displacement of the representative point in our diagram is governed by the equation

$$v\frac{dv}{d\theta_3}+K_1v+K_2\theta_3=C. \tag{16}$$

If v_1, v_2 denote the two values of v corresponding to a general value of θ_3 between α and $\alpha+\beta$, v_1 corresponding to the arc AB and v_2 to the arc BC, we have from (16)

$$\frac{1}{2}\frac{d}{d\theta_3}(v_1^2-v_2^2)+K_1(v_1-v_2)=0, \tag{17}$$

and hence we find

$$AE^2 - CE^2 = 2K_1 \int_{\alpha}^{\alpha+\beta} (v_1 - v_2) d\theta_3. \tag{18}$$

Since $v_1 - v_2 > 0$, we have $AE > CE$, where AE, CE represent positive numbers. When the burner comes into operation at C, and the system describes the arc CDA', it can be shown in precisely similar manner that $CE > A'E$. Therefore $AE > A'E$. Thus, in each complete oscillation, the points where the broken curve cuts $\theta_3 = \alpha$ approach the point E steadily, one from above and the other from below. These two sequences converge to limit points, which, obviously, either coincide at E or are equidistant from it. It is easy to see that the latter possibility must be ruled out. For let us suppose that there are limit points L, M such that $LE = ME$: the curve of the system passing through L in the positive sense will then cut ME internally. But in the history of the system there will be a curve which starts from a point on $\theta_3 = \alpha$ indefinitely close to L. If this initial point is sufficiently close to L, the curve will cut the lower half of $\theta_3 = \alpha$ above the point M. This is inconsistent with the hypothesis that M is the limit of the points on the lower half of $\theta_3 = \alpha$. Thus we see that the points of intersection with $\theta_3 = \alpha$ converge steadily to the point where that line cuts $v = 0$: in other words, *under ideal control the temperature of the room oscillates about the thermostatic temperature with steadily decreasing amplitude, and tends to it in the limit.*

Let us now consider the case of practical control ($\alpha_2 > \alpha_1$). Let us suppose that θ_3 increases up to α_2, and the burner ceases to operate at A in Fig. 5. The system then runs down the curve ABC, cutting $v = 0$ at B (say $\theta_3 = \alpha_2 + \beta$) and $\theta_3 = \alpha_1$ at C. The system then passes up CDA' cutting $v = 0$ at D ($\theta_3 = \alpha_1 - \delta$, say). For ABC we have the differential equation (12) with $\epsilon = 0$, and for CDA' the same equation with $\epsilon = 1$. Let us write

$$h = \alpha_2 - \alpha_1. \tag{19}$$

To each point A on the upper half of the line $\theta_3 = \alpha_2$ there corresponds a point A', obtained after a complete oscillation. From our theory of the ideal control, we know that, if the height of A above the line $v = 0$ is large compared with h, then A' will lie below A. On the other hand, if the distance of A from $v = 0$ is small compared with h, then a simple consideration of the general nature of the curves shows that A' will lie above A. Hence, as we move A down the line $\theta_3 = \alpha_2$, it overtakes its correspondent A', and thus there is a position of A (say $\bar{A}$) which coincides with its correspondent A'. Through this point there passes a closed curve, having corners where it meets $\theta_3 = \alpha_2$ and $\theta_3 = \alpha_1$: this represents a steady oscillation.

Now suppose the system starts at A, with a fairly large value of v, so that the corresponding A' lies below A. Following on the curve of the history after A', we get a spiral which (by the nature of the curves) can never cut itself. Thus the corner on $\theta_3 = \alpha_2$ moves steadily downwards: it cannot approach $v = 0$

indefinitely, because we have seen that it would commence to move up, which it cannot do, since the spiral does not cut itself. Thus the corner tends to the limit $\bar{A}$, and ultimately the motion of the system approaches asymptotically the closed circuit through $\bar{A}$. That, then, is the ultimate nature of the oscillation.

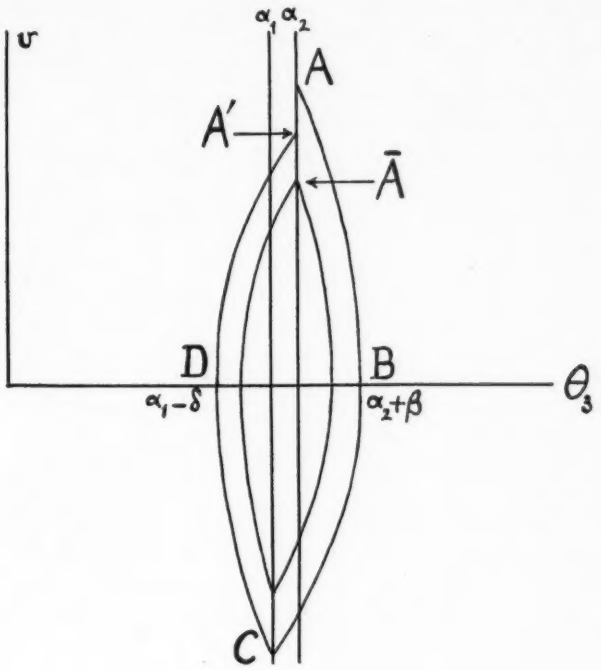


Fig. 5.

Let us assume that h is small; it is clear then that v_A will be small, when we approach the steady oscillation. We see from (18) that the loss of v-amplitude on passing from A to the point where ABC cuts $\theta_3=\alpha_2$ is of the order of β, which is of the order of v_A^2. There is a loss of the same order on passing from C along CDA' to the intersection with $\theta_3=\alpha_1$. In the steady oscillation this double loss of amplitude must be compensated by a gain in passing twice through the region $\alpha_1<\theta_3<\alpha_2$. This gain is of the order of

$$h\left(\frac{dv}{d\theta_3}\right)_A,$$

which, by (12), is seen to be of the order of h/v_A. Thus v_A^2 and h/v_A are of the same order. *Hence the v-amplitude of the steady oscillation is of the order of $h^{1/3}$, and hence the θ_3-amplitude is of the order of $h^{2/3}$.*

6. *Quantitative results concerning ultimate oscillations.*

For the case of ideal control, we wish to investigate (i) the limit of the periodic time of the small oscillations about the thermostatic temperature; (ii) the law of decay of amplitude of these oscillations.

For the case of practical control, we wish to investigate (i) the amplitude of the steady oscillation to which the system tends; (ii) the periodic time of this oscillation, and the parts of this time during which the burner is, and is not, in operation; (iii) the law of approach to the steady oscillation.

Up to a certain point, we can carry out these investigations in common, by treating the case of practical control ($h>0$), and then putting $h=0$ to get the case of ideal control.

We deduce from (12) the following values corresponding to $v=0$:

$$\frac{d\theta_3}{dv}=0,\quad \frac{d^2\theta_3}{dv^2}=\frac{1}{\epsilon K_3+C-K_2\theta_3},\quad \frac{d^3\theta_3}{dv^3}=\frac{2K_1}{(\epsilon K_3+C-K_2\theta_3)^2}. \tag{20}$$

If we write

$$L=K_2\alpha_2-C>0,\quad L'=K_3+C-K_2\alpha_1>0, \tag{21}$$

we have

$$\begin{cases} \text{on } ABC(\epsilon=0):\theta_3=\alpha_2+\beta-\frac{1}{2}\frac{v^2}{L+K_2\beta}+\frac{1}{3}\frac{K_1v^3}{(L+K_2\beta)^2}+\cdots, \\ \text{on } CDA'(\epsilon=1):\theta_3=\alpha_1-\delta+\frac{1}{2}\frac{v^2}{L'+K_2\delta}+\frac{1}{3}\frac{K_1v^3}{(L'+K_2\delta)^2}+\cdots. \end{cases} \tag{22}$$

Applying these equations to the points (A, C) and (C, A'), we have

$$\begin{cases} \beta=\frac{1}{2}\frac{v_A^2}{L+K_2\beta}-\frac{1}{3}\frac{K_1v_A^3}{(L+K_2\beta)^2}+\cdots, \\ h+\beta=\frac{1}{2}\frac{v_C^2}{L+K_2\beta}-\frac{1}{3}\frac{K_1v_C^3}{(L+K_2\beta)^2}+\cdots, \\ \delta=\frac{1}{2}\frac{v_C^2}{L'+K_2\delta}+\frac{1}{3}\frac{K_1v_C^3}{(L'+K_2\delta)^2}+\cdots, \\ h+\delta=\frac{1}{2}\frac{v_{A'}^2}{L'+K_2\delta}+\frac{1}{3}\frac{K_1v_{A'}^3}{(L'+K_2\delta)^2}+\cdots. \end{cases} \tag{23}$$

Now β, δ are of the order of v_A^2, and hence their omission from the denominators causes only an error in terms of the order of v_A^4. Hence, to the order of v_A^3, we have

$$(24)\qquad \begin{cases} 2L\beta = v_A^2 - \dfrac{2}{3}\dfrac{K_1}{L}v_A^3, & 2Lh + 2L\beta = v_C^2 - \dfrac{2}{3}\dfrac{K_1}{L}v_C^3, \\ 2L'\delta = v_C^2 + \dfrac{2}{3}\dfrac{K_1}{L'}v_C^2, & 2L'h + 2L'\delta = v_{A'}^2 + \dfrac{2}{3}\dfrac{K_1}{L'}v_{A'}^3. \end{cases}$$

Hence we deduce

$$(25)\qquad v_{A'}^2 + \frac{2}{3}\frac{K_1}{L'}v_{A'}^2 = 2L'h + v_C^2 + \frac{2}{3}\frac{K_1}{L'}v_C^3,$$

or, remembering that h is of the order of v_A^3, and $v_A > 0$, $v_C < 0$, $v_{A'} > 0$,

$$(26)\qquad v_{A'}^2 = 2L'h + v_C^2 + \frac{4}{3}\frac{K_1}{L'}v_C^3.$$

Also

$$(27)\qquad v_C^2 = 2Lh + v_A^2 - \frac{4}{3}\frac{K_1}{L}v_A^3,$$

and hence

$$(28)\qquad v_{A'}^2 = 2h(L + L') + v_A^2 - 4K_1(L'^{-1} + L^{-1})v_A^3/3,$$

which gives, for the decrease of amplitude in an oscillation,

$$(29)\qquad -dv_A = v_A - v_{A'} = \tfrac{2}{3}K_1(L'^{-1} + L^{-1})v_A^2 - h(L + L')v_A^{-1}.$$

For the steady oscillation, $dv_A = 0$: this gives for the v-amplitude ($\bar{v}$) of the steady oscillation, approximately,

$$(30)\qquad \bar{v}^3 = \frac{3h}{2K_1}\frac{L + L'}{L^{-1} + L'^{-1}} = \frac{3h}{2K_1}LL',$$

and for the corresponding θ_3-amplitudes, by (24),

$$(31)\qquad \beta = \frac{\bar{v}^2}{2L} = \frac{1}{2L}\left(\frac{3h}{2K_1}LL'\right)^{2/3}, \quad \delta = \frac{\bar{v}^2}{2L'} = \frac{1}{2L'}\left(\frac{3h}{2K_1}LL'\right)^{2/3}.$$

As for the periodic time of the steady oscillation, or an oscillation adjacent thereto, we have

$$(32)\qquad t(ABC) = \int_A^C (d\theta_3/v), \quad t(CDA') = \int_C^{A'} (d\theta_3/v).$$

In the former we use the approximate value given by (22),

$$(33a)\qquad d\theta_3 = -vdv/L,$$

and in the latter

$$(33b)\qquad d\theta_3 = vdv/L'.$$

Thus

$$t(ABC) = (v_A - v_C)/L, \quad t(CDA') = (v_{A'} - v_C)/L'. \tag{34}$$

But we have approximately

$$v_A - v_C = 2v_A, \quad v_{A'} - v_C = 2v_A, \tag{35}$$

so that we may write

$$t(ABC) = 2v_A/L, \quad t(CDA') = 2v_A/L'. \tag{36}$$

Thus in the steady oscillation the periodic time is

$$\tau = 2(L^{-1} + L'^{-1})\bar{v}, \tag{37}$$

and

$$\frac{t(\bar{A}BC)}{t(CD\bar{A})} = \frac{L'}{L}. \tag{38}$$

It will be well to restate the results obtained in an interpretable form:

In the case of practical thermostatic control, with the operating temperatures α_1, α_2, *where* $h(=\alpha_2-\alpha_1)$ *is positive but small, the ultimate state is one of steady oscillation through a temperature range of the order of* $h^{2/3}$. *The ratio of the temperature amplitudes above and below the thermostatic range is approximately*

$$\frac{\beta}{\delta} = \frac{L'}{L} = \frac{\bar{\theta} - \alpha}{\alpha - \theta_4}, \tag{39}$$

where θ_4 *is the (steady) temperature of the external atmosphere,* $\bar{\theta}$ *the maximum temperature to which the room can be raised by continuous operation of the burner, and* α *is the thermostatic temperature (that is, the mean of the approximately equal temperatures* α_1, α_2). *These amplitudes are*

$$\begin{cases} \beta = \dfrac{1}{2K_2(\alpha - \theta_4)}\left\{\dfrac{3hK_2^2}{2K_1}(\alpha - \theta_4)(\bar{\theta} - \alpha)\right\}^{2/3}, \\ \delta = \dfrac{1}{2K_2(\bar{\theta} - \alpha)}\left\{\dfrac{3hK_2^2}{2K_1}(\alpha - \theta_4)(\bar{\theta} - \alpha)\right\}^{2/3}, \end{cases} \tag{40}$$

where K_1, K_2 *are constants of the system, defined in* (5).

In a steady oscillation the ratio of the time during which the burner is idle to that during which it is in operation is

$$\frac{t(\bar{A}BC)}{t(CD\bar{A})} = \frac{L'}{L} = \frac{\bar{\theta} - \alpha}{\alpha - \theta_4}, \tag{41}$$

the same as the ratio of the amplitudes, and the periodic time of the steady oscillation is

$$\tau = 2(\bar{\theta} - \theta_4)\left\{\frac{3h}{2K_1K_2(\alpha - \theta_4)^2(\bar{\theta} - \alpha)^2}\right\}^{1/3}. \tag{42}$$

Stress may perhaps be laid on the result concerning the relative times of operation and rest of the burner on account of its simplicity. Let us point out its practical implications. Let us suppose that the external temperature is 10° and the thermostatic temperature is 70°: let us suppose that the burner in continuous operation would raise the room to 100°. (Actually, the water in the radiators would probably boil under continuous action of the burners: it is not essential to our theory that this "maximum possible" temperature should be physically realizable: it is merely a function of the constants of the system.) In this case the time during which the burner operates is twice the time during which it is idle, the ratio being (70–10)/(100–70). The temperature $\bar{\theta}$, it will be remembered, is given by

$$\bar{\theta} = \frac{K_3 + C}{K_2} = \frac{K_3}{K_2} + \theta_4;$$

here K_2, K_3 are permanent constants of the system. Thus

$$\bar{\theta} - \theta_4 = \frac{K_3}{K_2}, \tag{43}$$

a permanent constant of the system: it can therefore be found by one observation. Thus if on any occasion the values of the external temperature (θ_4) and the thermostatic temperature (α) are noted, and also the ratio of the times during which the burner is idle and operative respectively, we can compute from (41) the value of $\bar{\theta}$, and hence from (43) the value of K_3/K_2. We are then able to predict from (41) the ratio of times of idleness and operation for any assigned external temperature θ_4; the formula can be written in the form

$$\frac{t(\bar{A}BC)}{t(DC\bar{A})} = \frac{K_3}{K_2} \cdot \frac{1}{\alpha - \theta_4} - 1, \tag{44}$$

which, naturally, tends to infinity as θ_4 tends to α.

In the case of ideal control, we have seen that the amplitude of the oscillations tends to zero: it follows from (42), on putting $h = 0$, that *the periodic time also tends to zero.*

It remains to answer the questions raised regarding the decay of the oscillations. Let us consider first the case of practical control ($h > 0$). Writing in (29)

$$v_A = \bar{v}(1 + \xi), \tag{45}$$

where ξ is small, we have

$$\begin{aligned} -d\xi &= \{(4/3)K_1\bar{v}(L^{-1} + L'^{-1}) + h\bar{v}^{-2}(L + L')\}\xi \\ &= 3h\bar{v}^{-2}(L + L')\xi, \end{aligned} \tag{46}$$

for the decrease of ξ in a time τ, which is given by (37). Hence the law of decay is

$$\frac{d\xi}{dt} = -\frac{3h}{2\bar{v}^3}LL'\xi = -K_1\xi. \tag{47}$$

Thus ξ decreases according to the law

$$\xi = (\xi)_{t=0}e^{-K_1t}, \tag{48}$$

and v_A according to the law

$$v_A = \bar{v} + (v_A - \bar{v})_{t=0}e^{-K_1t}. \tag{49}$$

In the case of the ideal control ($h=0$), we return to (29), now in the form

$$-dv_A = \tfrac{2}{3}K_1(L^{-1} + L'^{-1})v_A^2, \tag{50}$$

this giving the decrease in v_A in time τ, given by (36) as

$$\tau = 2(L^{-1} + L'^{-1})v_A. \tag{51}$$

Thus v_A decays according to the law

$$\frac{dv_A}{dt} = -\tfrac{1}{3}K_1v_A, \quad v_A = (v_A)_{t=0}e^{-K_1t/3}, \tag{52}$$

and, by (29), β and δ decay according to the law

$$\beta = (\beta)_{t=0}e^{-2K_1t/3}, \quad \delta = (\delta)_{t=0}e^{-2K_1t/3}. \tag{53}$$

It is interesting to note that the rate of decay (in one case, towards steady oscillation and, in the other case, towards steady temperature) depends only on the constant K_1, given by (5). The greater K_1, the more rapid the decay. This is very natural in the mechanical analogue, since the frictional resistance is proportional to K_1. In the thermal problem, we see that if we are to increase the rate of decay, we are to increase k_{23}, k_{32}, k_{34}—one or more of them: k_{21}, which determines the rate at which the burner supplies heat to the water, has, strangely enough, nothing to do with it. Since k_{23}, k_{32} determine the rate at which heat flows from the radiators into the room, it is natural to expect an increase of rate of decay by increasing them. But it is strange to find that the rate of decay is increased by increasing k_{34}, that is, by decreasing the insulation between the room and the external atmosphere.

A CONVERGENCE TEST AND A REMAINDER THEOREM

By W. F. LIBBY, Berkeley, California

The specification of the terms of an infinite series necessarily involves a correspondence between the values and the ordinal numbers of the terms of the series. This paper is concerned with series for which this correspondence is an analytic one.

We define a *regular* series to be a series $\sum_{n=1}^{\infty} u(n)$ where $u(z)$ is a function of z which is analytic at infinity. Then, with each regular series, there is associated a set of constants $c_0, c_1, c_2, \cdots$ and a least index $L \geqq 1$ such that

$$u(n) = c_0 + \frac{c_1}{n} + \frac{c_2}{n^2} + \frac{c_3}{n^3} + \cdots \qquad n \geqq L; \tag{1}$$

and conversely any series $\sum u(n)$ for which (1) holds is a regular series.

THEOREM 1. *If $\sum u(n)$ is a regular series, then*

$$\sum_{n=L+1}^{\infty} \left[u(n) - c_0 - \frac{c_1}{n} \right] = \sum_{n=L+1}^{\infty} \sum_{k=2}^{\infty} \frac{c_k}{n^k}$$

where the series on the left is an absolutely convergent simple series and the series on the right is an absolutely convergent double series.

To prove this theorem, we note first that the series $c_0+c_1L^{-1}+c_2L^{-2}+\cdots$ converges and hence has uniformly bounded terms. Thus there is a constant M such that $|c_k|L^{-k} < M$ or $|c_k| < ML^k$, $k = 0, 1, 2, \cdots$. Therefore when $n > L$

$$\frac{|c_2|}{n^2} + \frac{|c_3|}{n^3} + \cdots < M\left\{\left(\frac{L}{n}\right)^2 + \left(\frac{L}{n}\right)^3 + \cdots\right\}$$

$$= M\frac{n}{n-L}\left(\frac{L}{n}\right)^2 \leqq M(L+1)L^2\frac{1}{n^2}$$

and absolute convergence of the double series in the right member follows from convergence of the series $\sum n^{-2}$. With this fact established, absolute convergence of the series in the left member and equality of the sums follow from well known theorems.

From Theorem 1 we obtain immediately the formula

$$\sum_{n=1}^{\infty} \left[u(n) - c_0 - \frac{c_1}{n} \right] = \sum_{n=1}^{L} \left[u(n) - c_0 - \frac{c_1}{n} \right] + \sum_{n=L+1}^{\infty} \sum_{k=2}^{\infty} \frac{c_k}{n^k}$$

and the following corollary: *In order that a regular series may be convergent, it is necessary and sufficient that $c_0 = c_1 = 0$.*

We now give a theorem which indicates the number of terms of certain regular series which it is sufficient to take to ensure that the remainder shall be less than a specified term.

THEOREM 2. *Any convergent regular series for which the coefficients c_n are real and non-negative has a value S given by*

$$S = \sum_{n=1}^{G+h} u(n) + \theta u(G) \tag{2}$$

where G is any integer $\geqq L$, h is any integer $\geqq G^2 - G$, and θ is a real constant depending on G and h for which $0 < \theta < 1$.

The theorem being trivial when $c_n=0$ for all n, we can suppose in our proof that $c_n\neq 0$ for at least one value of n; then $u(n)>0$ for all $n\geqq L$. Evidently $S>\sum_{n=1}^{G+h}u(n)$. Hence we can complete the proof of Theorem 2 by showing that

$$S<\sum_{n=1}^{G+h}u(n)+u(G). \tag{3}$$

But (3) is equivalent to the inequality $\sum_{n=G+h+1}^{\infty}u(n)<u(G)$ which, since convergence of $\sum u(n)$ implies $c_0=c_1=0$, can be written

$$\sum_{k=2}^{\infty}c_k\left(\sum_{n=G+h+1}^{\infty}\frac{1}{n^k}\right)<\sum_{k=2}^{\infty}c_k\left(\frac{1}{G^k}\right). \tag{4}$$

To establish (4) it is, since $c_k\geqq 0$, sufficient to show that

$$\sum_{n=G+h+1}^{\infty}\frac{1}{n^k}<\frac{1}{G^k}\qquad k=2,3,4,\cdots. \tag{5}$$

On multiplying (5) by G^k, we see that (5) holds for $k=2, 3, 4, \cdots$ when and only when it holds for $k=2$. Hence it will be sufficient to show that

$$\sum_{n=G+h+1}^{\infty}\frac{1}{n^2}<\frac{1}{G^2}. \tag{6}$$

But

$$\sum_{n=G+h+1}^{\infty}\frac{1}{n^2}<\int_{G+h}^{\infty}\frac{1}{x^2}dx=\frac{1}{G+h}.$$

Hence for (6) and therefore (3), it is sufficient to have $1/(G+h)\leqq 1/G^2$ or $h\geqq G^2-G$ and Theorem 2 is proved.

An interesting special case of Theorem 2 is that for which $G=1$; then we may take $h=0$ and $S\leqq 2u(1)$ follows.

Let us apply Theorem 2 to the following series with $L=1$

$$\sum_{n=1}^{\infty}\frac{8}{(4n-3)(4n-1)}$$

and check the result with the known sum, π. Taking $G=2$ and $h=4$, we find

$$S=\pi=8\left[\frac{1}{3}+\frac{1}{35}+\frac{1}{99}+\frac{1}{195}+\frac{1}{223}+\frac{1}{483}+\frac{\theta}{195}\right]$$

so that $3.1416=3.184+8[\theta/195]$ or $\theta=.665$ approximately; thus $0<\theta<1$ as the theorem states.

The author is much indebted to Professor Agnew of Cornell University for valuable suggestions as to shortening and simplifying the paper. The proof of Theorem 2 was especially so simplified.

POWER SUMS OF ROOTS

By H. D. GROSSMANN, De Witt Clinton High School, New York City

Let $r_1, r_2, \cdots r_n$ be the n roots of the equation $f(x) \equiv x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n = 0$, so that $a_h (h = 1, 2, \cdots, n)$ is the sum of their products taken h at a time; and let s_k be the sum of their kth powers, where k is any number, not necessarily integral or even real.

The relation

$$s_w - a_1 s_{w-1} + a_2 s_{w-2} - \cdots + (-1) a_w w = 0$$

for w any positive integer $\leqq n$, is well known. Since $s_0 = n$, for $w = n$ this may be written more symmetrically

$$s_n - a_1 s_{n-1} + a_2 s_{n-2} - \cdots + (-1) a_n s_0 = 0.$$

More generally

$$F(w) \equiv s_w - a_1 s_{w-1} + a_2 s_{w-2} - \cdots + (-1)^n a_n s_{w-n}$$

is identically zero *for any* w. For let $F_i(w) = r_i^w - a_1 r_i^{w-1} + a_2 r_i^{w-2} - \cdots + (-1)^n a_n r_i^{w-n}$, so that $F(w) = \sum_{i=1}^{n} F_i(w)$. Then $F_i(w) = r_i^{w-n} f(r_i) = 0$ since r_i is a root of $f(x) = 0$. Therefore $F(w) = 0$.

Power Sums of Roots of the Cubic $x^3 + a_2x - a_3 = 0$.

If $a_1 = s_1 = 0$, then $s_2 = a_1s_1 - 2a_2 = -2a_2$; $s_3 = a_1s_2 - a_2s_1 + 3a_3 = 3a_3$; $s_4 = a_1s_3 - a_2s_2 + a_3s_1 = -a_2s_2 = s_2^2/2$. In general, the relation between s_2, s_3, and sums of higher integral powers of the roots assumes a strikingly simple and elegant form, expressed in the

Theorem: If $n = 3$ and $a_1 = s_1 = 0$, and w is a positive integer > 3; then

$$s_w = w \sum \frac{(p_2 + p_3 - 1)!}{p_2!p_3!} \left(\frac{s_2}{2}\right)^{p_2} \left(\frac{s_3}{3}\right)^{p_3},$$

where the summation is over all non-negative integral values of p_2 and p_3 such that $w = 2p_2 + 3p_3$. (It is evident that $p_2 + p_3 \geqq 1$.)

For example, for $w = 4$ the only possible values for p_2 and p_3 are $p_2 = 2$, $p_3 = 0$. Hence we have

$$s_4 = 4 \frac{1!}{2!0!} \left(\frac{s_2}{2}\right)^2 = \frac{s_2^2}{2},$$

a result already noted.

To prove the theorem, consider all partitions of w of the form

$$w = 2 + 2 + \cdots + 2 + 3 + 3 + \cdots + 3.$$

Permute the terms of each partition in all possible ways. Let A be the class of those permutations whose final term is 2, B the class of those whose final term is

3. For example for $w=8$, we have A: 2222, 332; B: 233, 323. For a given partition containing p_2 2's and p_3 3's, the number of permutations in A, keeping the final 2 fixed, is

$$\frac{(p_2+p_3-1)!}{(p_2-1)!p_3!};$$

and the number in B, keeping the final 3 fixed, is

$$\frac{(p_2+p_3-1)!}{p_2!(p_3-1)!}.$$

Since $a_1=0$, $s_w=-a_2s_{w-2}+a_3s_{w-3}$. Then s_{w-2} and s_{w-3} may themselves be similarly decomposed, and by successive reduction we ultimately obtain

$$s_w = \sum_A(-a_2)^{p_2-1}(a_3)^{p_3}s_2 + \sum_B(-a_2)^{p_2}(a_3)^{p_3-1}s_3.$$

But $-a_2=s_2/2$ and $a_3=s_3/3$, and hence

$$\begin{aligned} s_w &= \sum_A\left(\frac{s_2}{2}\right)^{p_2-1}\left(\frac{s_3}{3}\right)^{p_3}s_2 + \sum_B\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3-1}s_3 \\ &= \sum_A\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}2 + \sum_B\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}3 \\ &= \sum\left[\frac{(p_2+p_3-1)!2}{(p_2-1)!p_3!} + \frac{(p_2+p_3-1)!3}{p_2!(p_3-1)!}\right]\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}. \end{aligned}$$

And since $2p_2+3p_3=w$, we have finally

$$s_w = w\sum\frac{(p_2+p_3-1)!}{p_2!p_3!}\left(\frac{s_2}{2}\right)^{p_2}\left(\frac{s_3}{3}\right)^{p_3}.$$

For example we find:

$$s_5 = 5s_2s_3/6, \qquad s_6 = s_2^3/4 + s_3^2/3,$$

$$s_7 = 7s_2^2s_3/12, \qquad s_8 = s_2^4/8 + 4s_2s_3^2/9,$$

$$s_9 = 3s_2^3s_3/8 + s_3^3/9, \qquad s_{10} = s_2^5/16 + 5s_2^2s_3^2/12,$$

$$s_{11} = 11s_2^4s_3/48 + 11s_2s_3^3/54, \qquad s_{12} = s_2^6/32 + s_2^3s_3^2/3 + s_3^4/27,$$

$$s_{13} = 13s_2^5s_3/96 + 13s_2^2s_3^3/54, \qquad s_{14} = s_2^7/64 + 35s_2^4s_3^2/144 + 7s_2s_3^4/81,$$

$$s_{15} = 5s_2^6s_3/64 + 25s_2^3s_3^3/108 + s_3^5/81, \text{ etc.}$$

It may be noted that these relations hold for $n>3$ if $s_1=a_1=a_4=a_5=\cdots=a_n=0$, since to the roots of $x^3+a_2x-a_3=0$ we merely add $n-3$ zero roots to obtain those of $x^n+a_2x^{n-2}-a_3x^{n-3}=0$.

A Symbolic Expansion for a Power Sum

The relation

$$s_w = a_1s_{w-1} - a_2s_{w-2} + \cdots + (-1)^{n-1}a_ns_{w-n}$$

may be written

$$s_w^{(0)} = a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)},$$

where $s_w^{(k)}$ symbolically represents s_{w-k}; or, since w is a variable, in the equivalent form

$$s_w^{(k)} = a_1s_w^{(k+1)} - a_2s_w^{(k+2)} + \cdots + (-1)^{n-1}s_w^{(k+n)}. \tag{1}$$

Theorem: $s_w^{(0)}$ may be symbolically represented by

$$[a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)}]^p$$

where p is any positive integer, and the superscripts in parentheses operate as exponents and are assigned their meanings only after expansion.

This may be proved by induction. It is true for $p=1$. It only remains to show that $S^p=S^{p+1}$, where

$$S \equiv [a_1s_w^{(1)} - a_2s_w^{(2)} + \cdots + (-1)^{n-1}a_ns_w^{(n)}].$$

Expanding S^p we have

$$S^p = \sum \frac{p!}{p_1!p_2!\cdots} a_1^{p_1}(-a_2)^{p_2}\cdots s_w^{(p_1+2p_2+\cdots)}, \tag{2}$$

where the summation is for all sets of non-negative p_i such that $\sum p_i=p$.

$$\begin{aligned} S^{p+1} &= S^pS \\ &= \sum \frac{p!}{p_1!p_2!\cdots} a_1^{p_1}(-a_2)^{p_2}\cdots s_w^{(p_1+2p_2+\cdots)}S, \end{aligned} \tag{3}$$

where the expansion is to be completed. But by (1), $s_w^{(p_1+2p_2+\cdots)}S$ gives $s_w^{(p_1+2p_2+\cdots)}$ after expansion, so that the completion of (3) will yield the same result as (2).

As an example of the theorem, for $n=2$, $p=1$, we have $s_w=a_1s_{w-1}-a_2s_{w-2}$; but also for $n=2$, $p=2$, we have $s_w=a_1^2s_{w-2}-2a_1a_2s_{w-3}+a_2^2s_{w-4}$; etc.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions and Discussions in the Monthly is open to all forms of activity in collegiate mathematics, including the teaching of mathematics, except for specific problems, especially new problems which are reserved for the department of Problems and Solutions.

RECURSION RELATIONS IN CERTAIN EXPANSION COEFFICIENTS

By R. S. UNDERWOOD, Texas Technological College

In the expansions of $\cos(n\theta)$ and $\sin(n\theta)/\sin\theta$ in terms of descending powers of the cosine, there are simple recursion relations by which the coefficients for successive n's may be computed almost as easily as the triangle of binomial coefficients may be constructed. It is proposed here to derive these relations and incidentally to obtain some interesting identities involving binomial coefficients, as (9) and (10) below.

Expanding $(\cos\theta+i\sin\theta)^n=\cos(n\theta)+i\sin(n\theta)$, and equating real and imaginary coefficients, we get the well-known equation

$$(1)\quad \cos(n\theta) = \cos^n\theta - \binom{n}{2}\cos^{n-2}\theta\sin^2\theta + \binom{n}{4}\cos^{n-4}\theta\sin^4\theta - \cdots$$

$$(2)\quad \frac{\sin(n\theta)}{\sin\theta} = \binom{n}{1}\cos^{n-1}\theta - \binom{n}{3}\cos^{n-3}\theta\sin^2\theta + \binom{n}{5}\cos^{n-5}\theta\sin^4\theta - \cdots$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

If we express $\cos(n\theta)$ in terms of descending powers of the cosine, we have, from (1) and (2)

$$(3)\quad \cos(n\theta) = A_{n,0}\cos^n\theta - A_{n,1}\cos^{n-2}\theta + A_{n,2}\cos^{n-4}\theta - \cdots \pm A_{n,r}\cos^{n-2r}\theta \mp \cdots$$

$$(4)\quad \frac{\sin(n\theta)}{\sin\theta} = B_{n,0}\cos^{n-1}\theta - B_{n,1}\cos^{n-3}\theta + B_{n,2}\cos^{n-5}\theta - \cdots \pm B_{n,r}\cos^{n-2r-1}\theta \mp \cdots;$$

where

$$(5)\quad A_{n,0} = 1 + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1},$$

$$(6)\quad A_{n,r} = \binom{n}{2r} + \binom{r+1}{1}\binom{n}{2r+2} + \binom{r+2}{2}\binom{n}{2r+4} + \cdots,$$

$$(7)\quad B_{n,0} = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},$$

$$(8)\quad B_{n,r} = \binom{n}{2r+1} + \binom{r+1}{1}\binom{n}{2r+3} + \binom{r+2}{2}\binom{n}{2r+5} + \cdots.$$

Comparing $A_{n,r}$ with the same coefficient derived by another method in Loney's Trigonometry, we have

$$(9)\quad \binom{n}{2r} + \binom{r+1}{1}\binom{n}{2r+2} + \binom{r+2}{2}\binom{n}{2r+4} + \cdots = 2^{n-2r-1}\left[\binom{n-r}{r} + \binom{n-r-1}{r-1}\right]$$

which holds wherever the binomial coefficient is defined, including the case $n=2, r=1$ if we define

$$\binom{n}{0} = \binom{0}{0} = 1.$$

Similarly, we have, using $B_{n,r}$

$$(10)\quad \binom{n}{2r+1} + \binom{r+1}{1}\binom{n}{2r+3} + \cdots = 2^{n-2r-1}\binom{n-r-1}{r}.$$

THEOREM: $A_{n,r} = 2A_{n-1,r} + A_{n-2,r-1}$.

Proof:

$$\begin{aligned}\cos(n\theta) &= \cos\theta\cos(n-1)\theta - \sin\theta\sin(n-1)\theta \\ &= \cos\theta\cos(n-1)\theta - [(\tfrac{1}{2})\cos(n-2)\theta - (\tfrac{1}{2})\cos(n\theta)] \\ (11)\quad \therefore \cos(n\theta) &= 2\cos\theta\cos(n-1)\theta - \cos(n-2)\theta \\ &= 2\cos\theta[2^{n-2}\cos^{n-1}\theta - \cdots \pm A_{n-1,r}\cos^{n-2r-1}\theta \mp \cdots] \\ &\quad - [2^{n-3}\cos^{n-2}\theta - \cdots \mp A_{n-2,r-1}\cos^{n-2r}\theta \pm \cdots].\end{aligned}$$

Equating the coefficients of $\cos^{n-2r}\theta$ in (3) and (11) we have the recursion formula of the theorem.

Similarly, writing

$$\begin{aligned}\sin(n\theta) &= \sin\theta\cos(n-1)\theta + \cos\theta\sin(n-1)\theta \\ &= \sin\theta[2^{n-2}\cos^{n-1}\theta - \cdots \pm A_{n-1,r}\cos^{n-2r-1}\theta \mp \cdots \\ &\quad + \cos\theta\sin\theta[2^{n-2}\cos^{n-2}\theta - \cdots \pm B_{n-1,r}\cos^{n-2r-2}\theta \cdots],\end{aligned}$$

we have

THEOREM: $B_{n,r} = A_{n-1,r} + B_{n-1,r}$.

We may then easily write successive rows of the arrays

1					1			
2	1				2			
2^2	3				2^2	1		
2^3	8	1			2^3	4		
2^4	20	5			2^4	12	1	
2^5	48	18	1		2^5	32	6	
.	.	.	.	.	.	.	.	.

In the left array the upper element in each column is 1 and other elements of all columns except the first are found as illustrated by the examples: $8=2\cdot 3+2$, $5=2\cdot 1+3$, $18=2\cdot 5+8$, etc. In the right array, $4=1+3$, $12=4+8$, $6=1+5$, $32=12+20$, etc., where, in the first equation for example, 1 is above 4 and 1 and 3 are in corresponding positions in the two arrays.

From the sixth rows of the arrays, for example, we have

$$\cos(6\theta) = 2^5\cos^6\theta - 48\cos^4\theta + 18\cos^2\theta - 1,$$

$$\frac{\sin(6\theta)}{\sin\theta} = 2^5\cos^5\theta - 32\cos^3\theta + 6\cos\theta.$$

THE CUBIC $x^2+y^2+z^2-2xyz=1$

By A. S. MERRILL, University of Montana

The theory of multiple correlation suggests the cubic appearing as the title of this discussion. The formula for the coefficient of multiple correlation with three variables is, in the usual notation,

$$r_{0.12} = \sqrt{\frac{r_{01}^2 + r_{02}^2 - 2r_{01}r_{02}r_{12}}{1 - r_{12}^2}}.$$

When this coefficient assumes its limiting values, ± 1, we have the relation

$$r_{01}^2 + r_{02}^2 + r_{12}^2 - 2r_{01}r_{02}r_{12} = 1,$$

or, in terms of x, y, and z,

$$x^2 + y^2 + z^2 - 2xyz = 1. \tag{1}$$

It can be shown by methods which are used in any first course in analytic geometry that plane sections of this cubic surface include all of the conic sections.

I. *The intersection with each coordinate plane is a circle.* For, replacing one of the coordinates, say z, by 0 in (1) we have

$$x^2 + y^2 = 1.$$

II. *The intersection with a plane parallel to any one of the coordinate planes at a distance of* 1 *is a straight line.* For, substituting 1 for z we have

$$x^2 + y^2 - 2xy = 0;$$

that is,

$$x - y = 0.$$

III. *The intersection with such a plane when the distance is less than* 1 *is an ellipse.* For, substituting c for z we have

$$x^2 + y^2 - 2cxy = 1 - c^2.$$

Transforming by a 45° rotation we obtain

$$\frac{x^2}{1+c} + \frac{y^2}{1-c} = 1.$$

IV. *The intersection with such a plane when the distance is greater than* 1 *is an hyperbola.* The proof is exactly as in III. The resulting equation represents an hyperbola since in this case $1-c$ is negative.

V. *The intersection with a plane which bisects the angle formed by a pair of coordinate planes is a parabola cut by a straight line perpendicular to its axis.* If we rotate the entire surface through 45° about the x axis, equation (1) becomes

$$x^2 + y^2 + z^2 - xy^2 + xz^2 = 1. \tag{2}$$

One of the planes in question now coincides with the xy-plane. The equation of the intersection is found by substituting 0 for z in (2). This gives

$$x^2 + y^2 - xy^2 = 1,$$

or

$$(x - 1)(y^2 - 1 - x) = 0.$$

The straight line is easily identified as one of the lines of (II) above. The size, shape, and position of the parabola are evident from its equation. The substitution $y=0$ in (2) gives the intersection made by a corresponding perpendicular plane. Its equation is

$$(x + 1)(z^2 + x - 1) = 0.$$

Evidently the two parabolas are oppositely oriented.

End points of parallel axes of successive parallel ellipses of III and vertices of corresponding hyperbolas of IV lie upon these parabolas. The straight line is the degenerate conic between the ellipses and hyperbolas. On each parabola, the points near the vertex are end-points of *minor* axes; farther out, there are end-points of major axes; and finally, vertices of the hyperbolas.

If x, y, and z are limited to values not greater than 1 (as would be necessary in the correlation theory), equation (1) represents a closed surface. A fairly satisfactory model can be constructed from wall board. Laminae of elliptical shape fastened together will produce the required solid. One axis of these laminae varies from 0 to $2\sqrt{2}$ while the perpendicular axis varies from $2\sqrt{2}$ to 0. The successive values depend upon the thickness of the wall board. Configurations upon this closed surface include the three circles of I, the family of ellipses of II, the six (partial) parabolas of V, and a regular tetrahedron whose vertices are the points (1, 1, 1), (1, −1, −1), (−1, 1, −1) and (−1, −1, 1).

On a Certain Transformation of Infinite Series[1]

By J. A. Shohat, University of Pennsylvania.

Introduction. The actual evaluation of the sum of a given convergent series is very often greatly hampered, if not rendered impossible, by the slowness of the convergence (example: $\pi/4=1-1/3+1/5-1/7+\cdots$). Euler, Markoff and others[2] have given practical methods for improving the convergence, which, however, are omitted even in many advanced courses on infinite series. Thus, it seems worthy of interest to give an extremely simple transformation of infinite series which, in case of convergence, often renders it more rapid, and thus the series in question becomes more useful for practical applications. In case of divergence, the same transformation often yields the asymptotic expression (for $n\to\infty$) of S_n—sum of its first n terms. We give concrete illustrations for both cases, for the latter by means of Stirling's formula.

I. *Transformation of S_n.* The transformation referred to above is:

$$S_n \equiv \sum_{i=1}^{n} u_i = \sum_{i=1}^{n-1} i(u_i - u_{i+1}) + nu_n, \tag{1}$$

an identity holding true for any $u_1, u_2, \cdots, u_n$ (special case of Abel's transformation). Thus we are led to consider two series:

$$(U)\ \sum_{n=1}^{\infty} u_n, \quad (V)\ \sum_{n=1}^{\infty} v_n \qquad (v_n = n(u_n - u_{n+1})).$$

We know that, if in the series $\sum_{n=1}^{\infty} a_n$, $\lim_{n=\infty} na_n$ exists and $\neq 0$, then the series diverges (the converse is not true). This, in conjunction with (1), leads at once to several interesting conclusions.

Theorem I. *If $\lim_{n=\infty} nu_n = l$ exists, then the two series (U) and (V) are either both convergent or both divergent, according as $l=0$ or $l\neq 0$. In case of convergence, the two series have the same sum.*

---

[1] Read before the Pennsylvania Chapter of Pi Mu Epsilon, in October, 1932.

[2] Cf. K. Knopp, *Theorie und Anwendungen der Unendlichen Reihen*, 2nd ed. (1924), pp. 242–249.

THEOREM II. *If* (U) *and* (V) *both converge, then* $\lim_{n=\infty} nu_n = 0$; *hence* (U) *and* (V) *have the same sum.*

It is the latter property which makes the transformation (1) very valuable as the illustrations below indicate. The following theorem, not directly connected with the main line of the present discussion, is also of frequent use in the theory of infinite series.

THEOREM III. *If* $\{u_n\}$ *is a monotonic decreasing sequence and* (U) *is convergent, then necessarily* $\lim_{n=\infty} nu_n = 0$.

In fact, here (V) consists of positive terms; hence, it either converges or diverges to $+\infty$. In the first case our statement is proved by Theorem I; in the second case, (1) shows, since (U) converges, that $\lim_{n=\infty} nu_n = -\infty$, which is impossible.

2. *Illustrations in case of convergence.* (i) Consider the series

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}. \tag{2}$$

Here

$$u_n = \frac{1}{n^2}, \quad v_n = n\left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) = \frac{2}{(n+1)^2} + \frac{1}{n(n+1)^2}$$

$$\frac{\pi^2}{6} = 2\sum_{n=1}^{\infty} \frac{1}{(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = 2\sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} - 2$$

$$\frac{\pi^2}{6} = 2\cdot\frac{\pi^2}{6} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} - 2$$

$$\frac{\pi^2}{6} = 2 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}. \tag{3}$$

If we apply once more the transformation (1), we get:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = 3\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2}$$

$$= 3\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2} - \frac{3}{4}$$

$$\frac{\pi^2}{6} = \frac{13}{8} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(n+1)^2(n+2)^2}. \tag{4}$$

We thus have transformed (2) *into more rapidly convergent series* (3) *and* (4), the remainders, i.e. the sums of all terms starting with the $(n+1)$st being respectively of order of magnitude n^{-1}, n^{-2}, n^{-3}. With little computation, we could repeat the same transformation again. We also notice that, using the series (3) *and* (4), we get numerical values *limiting* π *both from above and below.* Thus,

$$\frac{\pi^2}{6} > \frac{13}{8} + \frac{1}{2}\cdot\frac{1}{2^2\cdot 3^2}, \quad \frac{\pi^2}{6} < 2 - \frac{1}{4} - \frac{1}{2\cdot 9}$$

$$\frac{1}{6}\sqrt{354} < \pi < \frac{1}{6}\sqrt{366}; \quad 3.13\cdots < \pi < 3.18\cdots . \tag{5}$$

Hence, taking one term in (4) and two terms in (3), we obtain the value of π with an error not exceeding 0.05. It can be shown (making use of $\int_1^n x^{-2}dx$) that

$$\lim_{n=\infty} n\left[\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots\right] = 1.$$

It follows that about *twenty* terms are needed in (2), if we wish to obtain π with the above accuracy.

(ii) Apply the transformation (1) to

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \pm \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}. \tag{6}$$

We get, if n is even: $n = 2m$,

$$S_{2m} = \sum_{i=1}^{2m-1}(-1)^{i-1}\left[1 + \frac{1}{4i^2-1}\right] - \frac{2m}{4m-1},$$

and a similar formula for S_{2m+1}. Letting $m \to \infty$, we thus obtain:

$$\frac{\pi}{4} = \frac{1}{2} + \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} \mp \cdots = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{4n^2-1}. \tag{7}$$

The series (7), being an alternating one, yields both upper and lower bound for for π. It converges far more rapidly than (6). If, for example, an accuracy of 0.01 for $\pi/4$ is desired, we need 50 terms in (6), and only 5 terms in (7).

3. *Illustration: some elementary summations.* It is, we believe, of some interest to show that the same transformation (1) yields, in many cases, with very little or no computation, the *precise expression* for certain $S_n = \sum_{i=1}^{n} u_i$. As an example, the arithmetic series $\sum_{i=1}^{n} i$ may serve. Here, by (1),

$$\sum_{i=1}^{n} i = -\sum_{i=1}^{n-1} i + n^2 = -S_{n-1}^{(1)} + n^2 = -S_n^{(1)} + n^2 + n; \; S_n^{(1)} = \frac{n(n+1)}{2}.$$

Similarly,

$$S_n^{(2)} \equiv \sum_{i=1}^{n} i^2 = -\sum_{i=1}^{n-1} i(2i+1) + n^3 = -2S_n^{(2)} - S_{n-1}^{(1)} + n^3 + 2n^2$$

$$S_n^{(2)} = \frac{1}{3}\left[n^3 + 2n^2 - \frac{(n-1)n}{2}\right] = \frac{n(n+1)(2n+1)}{6}.$$

In the same way we can evaluate $S_n^{(3)} \equiv \sum_{i=1}^{n} i^3, S_n^{(4)}, \cdots$. In general, if the precise expression of $\sum_{i=1}^{n} u_i$ is known, (1) gives at once the expression for

$\sum_{i=1}^{n} i(u_i - u_{i+1})$. For example, applying (1) to $\sum_{i=1}^{n} q^i [= (1-q^{n+1})/(1-q)]$, we get immediately:

$$\sum_{i=1}^{n} iq^{i-1} = \frac{1 - q^{n+1}}{(1-q)^2} - \frac{(n+1)q^n}{1-q}.$$

Of course, we cannot expect such an elementary transformation as (1) to work successfully in all cases, as is shown by the series $\log 2 = 1 - 1/2 + 1/3 - 1/4 + \ldots$.

4. *Illustration in case of divergence.* We close our discussion by showing that

$$\lim_{n=\infty} \frac{n!}{n^{n+1/2}e^{-n}} \text{ exists, } = l, \text{ with } 0 < l < \infty. \tag{8}$$

[Stirling's formula gives for l the value $(2\pi)^{1/2}$]. Applying the transformation (1) to $\sum_{i=1}^{n} \log i$, we get:

$$\sum_{i=1}^{n} \log i = \sum_{i=1}^{n-1} i \log \frac{i}{i+1} + n \log n = -\sum_{i=1}^{n-1} i \log\left(1 + \frac{1}{i}\right) + n \log n. \tag{9}$$

Taking 3 terms in the expansion for $\log(1+x)$, we have:

$$\log\left(1 + \frac{1}{i}\right) = \frac{1}{i} - \frac{1}{2i^2} + \frac{\theta_i}{3i^3} \quad (i \geqq 1;\ 0 < \theta_i < 1),$$

which, substituted in (8), leads to

$$\sum_{i=1}^{n} \log i = n \log n - (n-1) + \frac{1}{2} \sum_{i=1}^{n-1} \frac{1}{i} - \frac{1}{3} \sum_{i=1}^{n-1} \frac{\theta_i}{i^2}. \tag{10}$$

Making use of the convergence of the series $\sum_{i=1}^{\infty} \theta_i/i^2$ and of the existence of $\lim_{n=\infty} (\sum_{i=1}^{n} 1/i - \log n)$ (which is readily established, by means of $\int_1^n dx/x$), we obtain from (10) immediately:

$$\lim_{n=\infty} \left(\sum_{i=1}^{n} \log i - n \log n - \tfrac{1}{2} \log n + n \right) \text{ exists, } = B, \text{ with } 0 < B < \infty,$$

which is but another way of writing (8).

RECENT PUBLICATIONS

EDITED BY ROGER A. JOHNSON, Brooklyn College of the City of New York

All books for review should be sent directly to the editor of this department, at Brooklyn College, 66 Court Street, Brooklyn, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Caractéristiques des systèmes différentiels et propagation des ondes. By Tullio Levi-Civita. Paris, Librairie Felix Alcan, 1932. x+114 pages. 20 francs.

Mathématiques financières. By J. Dubourdieu. Paris, Armand Colin, 1932. 220 pages. 10.50 francs; bound, 12 francs.

The Principles of Financial and Statistical Mathematics. By Maximilian Philip. The College of the City of New York Series in Commerce, Civics, and Technology. New York, Prentice-Hall, Inc., 1932. xx+406 pages. $3.50.

A First Course in Calculus. By E. S. Crawley and P. A. Caris. New York, F. S. Crofts & Co., 1933. x+342 pages.

Introductory College Algebra, Revised. By H. L. Rietz and A. R. Crathorne. xii+306+24 pages. $1.76.

Analytic Geometry. By Frederick S. Nowlan. New York, McGraw-Hill Book Company, 1933. xii+296 pages. $2.25.

Gli elemente d'Euclide e la critica antica e moderna. Edited by Federigo Enriques. Libro X. Bologna, Nicola Zanichelli, 1932. 326 pages.

Foundations of the Theory of Algebraic Numbers. By Harris Hancock. Volume II, The General Theory. New York, The Macmillan Company, 1932. xxviii +654 pages. $8.00.

C. A. Bjerknes, Sein Leben und seine Arbeit. By V. Bjerknes. German translation by Else Wegener-Koppen. Berlin, Julius Springer, 1933. vi+218 pages. RM 8.60.

Numerology. By E. T. Bell. Baltimore, Williams and Wilkins, 1933. viii+188 pages. $2.00.

Vector Analysis. By H. B. Phillips. New York, John Wiley, 1933. viii+236 pages. $2.50.

The Universe of Science. By H. Levy. New York, The Century Company, 1933. xvi+224 pages. $2.50.

Mathematical Excursions. By Helen A. Merrill. Norwood, Mass., The Norwood Press, 1933. xiv+146 pages.

Arithmetic for Teachers. By Harriet E. Glazier. New York, McGraw-Hill, 1932. xvi+292 pages. $2.00.

Elementary Mathematical Analysis. Volume II. By Mayme I. Logsdon. New York, McGraw-Hill, 1933. x+188 pages. $1.75.

Number, the Language of Science. By Tobias Dantzig. Second Edition, Revised. New York, Macmillan, 1933. xiv+262 pages. $2.50.

(According to the preface, the revision has been confined to the correction of errors, the recasting of a few passages, and the addition of a page of bibliographical notes. This book was reviewed in the Monthly in March, 1931, issue, page 164.)

Scripta Mathematica

A new periodical published by Yeshiva College, Amsterdam Avenue and 186th Street, New York, N. Y.

The mathematical world suffered a distinct loss when the publication of *Bibliotheca Mathematica* was suspended in 1915. Its editor, Gustav Enestrom,

shortly before his death, expressed the wish to David Eugene Smith that some way could be found to continue this journal in America. Acting on this suggestion, the Mathematical Association of America endeavored to secure a subsidy fund sufficient to insure the success over a period of years of such an undertaking. Failing in this attempt, the Association, late in 1928, undertook an international canvass to determine whether a sufficiently large advance subscription list could be obtained to guarantee the project. The result was very enthusiastic *moral* support but not enough subscription pledges to warrant going ahead. Possibly the impending world-wide financial crash was already casting its shadow before it.

But now, in the midst of the depression, Yeshiva College in New York City comes forward with financial support[1] for a new quarterly journal, *Scripta Mathematica*, "the first of a series of publications planned by the college" with the ideal of "learning for the sake of learning." This journal, under the editorship of Jekuthiel Ginsburg, professor of mathematics in Yeshiva College, is to be devoted "to the philosophy, history, and expository treatment of mathematics," and will be entirely international in its scope. High standards for this journal will be assured by the character of its editorial board which includes as associate editors, Raymond Clare Archibald, Cassius Jackson Keyser, Louis Charles Karpinski, Gino Loria, Vera Sanford, Lao Genevra Simons, and David Eugene Smith. Although it will be seen that this publication is not in any sense a continuation of the *Bibliotheca Mathematica*, which it is hoped will sometime be revived, it promises a series of historical articles which will be welcomed as supplementing those which frequently appear in the MONTHLY.

The two numbers already issued (September and December, 1932) show a wide variety of interest and appeal—to the scholar, to the casual reader in other fields as well as in mathematics, and to any intelligent person whose curiosity leads him to wonder about the difficulties encountered by the human race in developing such a fundamental subject as mathematics. Any light which can be thrown upon the age-long growing pains of mathematics by historical research is not only of interest on its own account but is likely to point the way by analogy to a sympathetic attitude toward the difficulties now experienced with mathematical studies by the children of the present generation. For this and many other reasons we welcome the advent in this country of a historical journal in the field of mathematics.

Space will not permit a critical review here of the many interesting topics discussed in these two numbers, but the reviewer hereby confesses to the surrender of his own cold and critical attitude and warmly recommends their careful reading to the whole constituency of this MONTHLY. Without prejudice to the other contributions, all of which seem very worthy, consider for example, the

[1] It goes without saying that Yeshiva College hopes for a liberal subscription list which, so far as possible, may diminish the subsidy required and thus release funds for other contemplated publications. The subscription price is three dollars per year, which is a very reasonable charge for a volume of more than 380 pages of high character and excellent makeup.

following random samples: "Gaspard Monge, Politician," by David Eugene Smith, an article based on Smith's manuscript collection now donated to Columbia University;[1] "The Meaning and the Bearings of Mathematics," two articles by Cassius Jackson Keyser, written in his own clear and convincing style so well known to the mathematical world: "A. L. Cauchy in the History of Analytic Geometry," by Gino Loria; "Jacob ben Machir's Version of Menelaus's Work on Spherical Trigonometry," a translation by Jekuthiel Ginsburg, parts (1) and (2) and to be continued;[2] "Rare Mathematical Books in the University of Michigan Library," by Louis C. Karpinski; "The Ancient Peruvian Abacus," by L. Leland Locke, author of the richly illustrated work on *The Ancient Quipu or Peruvian Knot Record*, published by the American Museum of Natural History; "The Concept of Infinity," by D. Mordukhai-Boltovskoy, the first of a series of historical notes on this topic, translated from the Russian by Ginsburg; "A German-American Algebra of 1837," by Lao Genevra Simons. All told there are twenty-one signed[3] articles in the two numbers, aside from several excellent Book Reviews and a Department of Notes and Queries conducted by Raymond Clare Archibald. Of this latter department the reviewer wishes to say that it had long been his desire to see such a department attached to, or connected with, this MONTHLY and edited by Archibald, than whom probably there is no one in America or in Europe who is better qualified to develop such a department. Already in the second number this department occupies eighteen pages, much of which is of absorbing interest. Under the heading, "*Bibliographia de Mathematicis*," Archibald has listed some forty names of mathematicians together with references concerning them to articles or books many of which would be difficult for the average reader to obtain or the existence of which in many cases might be entirely unknown to the casual observer. He proposes to extend this list and keep it as complete as possible in the succeeding issues of the journal. In this and many other ways this department will render a real service. Perhaps, after all, it is better so rather than as a supplement to the MONTHLY.

To Yeshiva College the reviewer would extend his congratulations, believing that the new journal will soon make an honorable name for itself in the mathematical world, and will fulfill a very useful mission.

H. E. SLAUGHT

Topology. By Solomon Lefschetz. American Mathematical Society Colloquium Publications, Vol. XII. New York, 1930. ix+410 pages. $4.50.

Since the appearance of Veblen's book[4] on topology in the Colloquium series

[1] This magnificent collection of original manuscript will afford a rich source of historical material for use of this journal for many years to come.

[2] In the first article Ginsburg's name, as translator and editor, is omitted and in the second article, (except in the Hebrew text) there is no indication that it is a continuation of the previous one, and, indeed, the two titles are not the same in the text though they are in the Table of Contents. This is an unfortunate confusion.

[3] Every article, long or short, even if only a note or query, is signed by the author, thus indicating complete individual responsibility.

[4] O. Veblen, *Analysis Situs;* now in second edition, 1931.

of the American Mathematical Society, there have appeared important new combinatorial theories, each of which relates to the manner in which, given a polygonal configuration or "complex" K, we define an algebraic *chain* of K. Of these the "mod m" theory was introduced by Alexander in 1926, and the "mod L" (where L is a subcomplex of K) and "mod (L, m)" by Lefschetz about 1927.

Along with the emergence of these new basic theories there have evolved a wider degree of application of the combinatorial method and a host of new and far-reaching results. As instances of these we have the application of the combinatorial method to the study of compact metric spaces; applications to algebraic geometry (whose topological side, for the case of functions of two or more variables, really owes its development to Lefschetz); the increasing generality of duality theorems and the investigation of their interrelations; extension of the topological theory of intersections of complexes; etc.

In view of the large amount of material that has thus accumulated since the appearance of the Veblen book, there would seem to be no question that a work such as the one under review will fill a distinct need, especially as it covers so completely the extensive literature. In the opinion of the reviewer it does not (and the author apparently did not intend that it should) displace the excellent introduction to the combinatorial method furnished by the Veblen Colloquium. To be sure, it is complete in itself, but it seems advisable for the beginner in combinatorial theory to read first the Veblen book (beginning as it does with the elementary treatment of the 1- and 2-dimensional cases and then proceeding to the n-dimensional case) and then to use the Lefschetz exposition as supplementary.

The book has an index and there is included a valuable bibliography. Numerous diagrams are given in the early part of the book. There seem to be few misprints; it is important to call attention, however, to the erroneous definition of "continuous transformation" on page 3, which apparently escaped detection in the proof reading.[1]

R. L. WILDER

Differential Equations from the Algebraic Standpoint. By J. F. Ritt. American Mathematical Society Colloquium Publications, Vol. XIV. New York, 1932. x+172 pages. $2.50.

Systems of differential equations, ordinary and partial, which are algebraic in the unknown quantities and their derivatives, furnish the subject of this book. Not much has been done heretofore with the algebraic theory of such systems. In their treatment it has been customary to assume certain canonical forms; but the methods which have been proposed for reduction to such forms are subject to limitations and contain no effective means for preventing the entrance of extraneous solutions. The theory of systems of algebraic equations is in a much more satisfactory state. The aim of the present book is "to bring to the theory

[1] See the list of corrections given in connection with the review by P. A. Smith, Bull. Amer. Math. Soc., vol. 37 (1931), in the footnote on pp. 647–648.

of systems of differential equations, which are algebraic in the unknowns and their derivatives, some of the completeness enjoyed by the theory of systems of algebraic equations." The point of view taken is that of the author's memoir in volume 32 of the *Transactions of the American Mathematical Society*. While the existing theory of algebraic manifolds has afforded a guide to the investigation, it has turned out, as one would have expected, that the essentially new phenomena involved have called for the development of new methods. While "many questions still remain for investigation," the author has nevertheless made a marked advance toward a satisfactory theory of algebraic differential equations.

R. D. CARMICHAEL

Das Hauptproblem der Äusseren Ballistik. By Kyrill Popoff. Leipzig, Verlagsgesellschaft M.B.H., 1932. xi+214 pages. RM 16.40.

From both the theoretical and also the experimental point of view, the subject of exterior ballistics is naturally divided into two parts:

(A) The translation of projectiles, considered simply as heavy particles, through a resisting medium (the atmosphere) while subject to the acceleration of gravity.

(B) The motions of projectiles considered as rotating bodies of appreciable dimensions.

From the theoretical point of view, the major difficulties presented by the subject of exterior ballistics are encountered in (B), for to the differential equations of translation are at once added the Eulerian equations that define the gyrations of projectiles and the conditions for their stability. From the experimental point of view, the problems of greatest difficulty and greatest value are also encountered in (B), for they include all problems relating to stability and accuracy of flight.

The problems included under (B) appear not to have been investigated, either theoretically or experimentally, until during the World War, and then only in England and the United States. The work of the English (Fowler, Gallop, Lock, and Richmond) is contained in Philosophical Transactions of the Royal Society of London (1920), and the theoretical part of the work done in America is set forth in my *New Methods in Exterior Ballistics* (The University of Chicago Press).

In spite of the fact that the book of Popoff appeared in 1932, it contains no reference whatever to the problems of ballistics included under (B).

Those parts of ballistics included under (A) are likewise of two kinds:

(a) The theory of the translation of projectiles under ideal conditions.

(b) The theory of the effects of abnormal conditions upon the flight of projectiles, and the development of methods of rapidly applying them.

Under (b) are included the theory of the effects of: (1) abnormal intitial velocities, (2) abnormal directions of fire, (3) abnormal weights of projectiles, (4) the direction and velocity of winds at various altitudes, (5) abnormal air

densities, (6) variations in the elasticity of the air, (7) variations in gravity, (8) the rotation of the earth, and (9) the curvature of the surface of the earth. In addition, there is included under (b) the theory of "weighting factors" by means of which the effects of abnormalities may be easily taken into account. Obviously, under modern conditions, these problems are of great practical importance, and theoretically they have been found to be of great interest. Yet Popoff's book contains no reference whatever to them.

It follows that Popoff treated only the flight of projectiles, considered as heavy particles moving under ideal conditions, which was the problem treated by Euler, Cauchy, Siacci, and other writers on ballistics up to the time of the World War. In fact, he reproduces most of the methods of earlier writers, though correcting their errors in logic and extending their results, often to a notable degree. Like earlier writers, he considers the problem first in an ideal homogeneous atmosphere, devoting nine chapters and 136 pages to this part. Then he treats the problem for an atmosphere decreasing in density with altitude, devoting six chapters and 72 pages to this part. Frequently, he follows earlier writers in resorting to various approximations to the functions involved in order to separate the variables in the differential equations, sometimes at the cost of using another parameter than the time as an independent variable, and consequently requiring an inversion to reduce the results to a form suitable for practical use. Yet he uses series in many instances, always with an attention to their convergence not found in the work of other Continental writers on ballistics.

From the practical point of view, the work of Popoff is of no particular significance. From the theoretical point of view, it nearly completes the work of those by whom he appears to have been influenced, and logically it is vastly superior to that of any of them. In this respect, it is really distinguished, though on page 14 he makes the strange remark that experiments indicate that the retardation as a function of the velocity is continuous and satisfies the Lipschitz condition. The Lipschitz condition includes continuity and obviously neither could ever be established by experiment. The property is *assumed*, and from that point on the logic is secure. There is no intention to make any capital of this slip, except as it illustrates the very slight connection of the author with the practical aspects of his problem, for this misstatement is quite exceptional in a work distinguished for its rigor in a field often filled with errors in reasoning.

F. R. MOULTON

Differential and Integral Calculus. By J. H. Neelley and J. I. Tracey. New York, The Macmillan Co., 1932. viii+496 pages. $4.00.

This text, by well-known authors, covers the usual material of an elementary course in the calculus, including a brief introduction to ordinary differential equations. The presentation is clear throughout and, except in a few instances, accurate. Difficult proofs are avoided, references being given to more advanced works. Care is taken to indicate what assumptions are made in the proofs given.

The reader will notice the following points. The expression for the differential of arc is obtained by considering the velocity of a point moving along the arc. In problems involving summat on leading to a definite integral, Duhamel's theorem or an equivalent device is avoided by the assumption that infinitesimals of higher order may be neglected. What is defined to be the double integral is really the iterated integral. The general solution of the linear second order differential equation with constant coefficients is not given, but the authors show how the solutions can be obtained in any particular problem by means of a substitution on the dependent variable. The proof of the fact that the general solution of a differential equation of order n contains n arbitrary constants is faulty since it implicitly assumes that the equation obtained by eliminating the constants must be the original equation. There are no applications of partial differentiation to space geometry.

The book contains three chapters which review plane and solid analytic geometry and which should prove very useful to the student. We must point out, however, that the asymptotes of the hyperbola are incorrectly obtained.

The typography and general appearance of the book are attractive. There are two obvious misprints which will cause no trouble (p. 333 and p. 435). Numerous worked out examples and complete sets of exercises follow each topic.

SAMUEL BOROFSKY

Mathematical Nuts. By Samuel I. Jones. Published by the author, Nashville, Tenn., 1932. x+340 pages. $3.50.

This book contains a variety of nuts for lovers of mathematics. The author informs his readers that there are many varieties of nuts and various ways of classifying them. He then quotes from the humorist, Ralph A. Parlette, a schoolmate of the reviewer, in his *It's Up to You*. "I hold in my hand this glass jar containing little white beans and big black walnuts. I mix them all up. Then I shake the jar. They un-mix. The walnuts go to the top and the little beans go to the bottom. This little jar is a picture of what is going on everywhere in this world all the time. The world is just a big jar of life. . . . All kinds of people are in the jar of life. The jar goes on shaking all the time. It never stops shaking. We are either Big Nuts or Little Nuts." It is hard to tell which is the more numerous. It depends upon one's point of view, we suppose.

The book contains nine sections: I Nuts for Young and Old; II Nuts for the Fireside; III Nuts for the Classroom; IV Nuts for the Math Club; V Nuts for the Magician; VI Nuts for the Professor; VII Nuts for the Doctor; VIII Nuts Cracked for the Weary; IX Nut Kernels.

Some years ago the author published his *Mathematical Wrinkles*, a book very favorably commended by educators and editors in both England and America.

In the preparation of *Mathematical Nuts*, the author has far overstepped his former efforts. The reviewer has never before seen anywhere such an array

of interesting, stimulating, and effort inducing material as is here brought together. The questions range from the very easy ones, such as "Express 3 by using three threes" to some very difficult ones requiring the Calculus.

Much valuable information may be gained by young and old alike in devoting some of their leisure time to cracking of these nuts.

B. F. FINKEL

Introduction to Trigonometry and Analytic Geometry. By Ernest Brown Skinner, New York, The Macmillan Company, 1932. xi+189 pages. $1.80.

The problem of shortening the road to the calculus for those students who enter college with some two and a half years of high school mathematics is one which concerns all college teachers. They must therefore be interested in the book under review which represents a carefully worked out contribution toward the solution of the problem.

An efficient and unified treatment covers the standard topics in Trigonometry and Analytic Geometry with sufficient brevity to bring them within the work of a single semester. Naturally, the necessary curtailment involves some sacrifices; for example, there is no occurrence of curves like the lemniscate and limaçon, nor is there any great variety of exercises in the geometry of three dimensions. Incidentally, while mentioning geometry of three dimensions, the reviewer would wish for the introduction and considerable illustration of the concept of "direction numbers," but perhaps this is only a matter of personal taste. The really essential material is adequately and often freshly treated, and exercises given in sufficient, yet not overwhelming, number. Flexibility and stimulation are provided by the treatment in the text of well-chosen applications to practical problems, and by a suggestive chapter (XII) on "Miscellaneous Topics," as well as by the occasional adoption of a sort of "higher standpoint" in the treatment of elementary matters.

The typographical errors, of which several were noted, are nearly all obvious, though the omission of parentheses in the bold-face formulas on page 64 might trouble students. And is it not a little dangerous to tell freshmen (page 171) that "homogeneous equations may be solved for the variables"? Perhaps the answer is that the author presupposes a *good* course in College Algebra, and has furthermore deliberately left a good deal to the teacher. In any case, adverse criticisms can be directed only at minor matters, and the text merits serious attention, especially in institutions where the pedagogical problem at whose solution it aims is acute.

B. P. GILL

MATHEMATICS CLUBS

EDITED BY F. M. WEIDA, The George Washington University, Washington, D. C.

All reports of club activities, suggestions and topics for club programs, and material of interest should be sent to F. M. Weida, The George Washington University, Washington, D. C. All manuscript should be typewritten with double spacing, and with margins at least one inch wide. All club activity manuscripts for the academic year 1932–1933 should be submitted for publication not later than June 1, 1933.

CLUB ACTIVITIES

1931–1932

THE PI MU EPSILON MATHEMATICAL FRATERNITY

Pi Mu Epsilon is an academic fraternity in institutions of university grade. Its primary aim is the advancement of mathematics and scholarship. It is a living, active, working fraternity of scholars in which the members are actively engaged in study and research and the exchange of ideas in the field of mathematical science.

The annual business meeting and banquet was held in the Hotel Morton in Atlantic City at noon on December 27, 1932. The attendance was seventeen, including the following sixteen members of the fraternity: J. H. Bushey, Marguerite D. Darkow, F. F. Decker, H. A. DoBell, M. Dresher, H. S. Everett, E. J. Finan, E. R. Hedrick, Louis Ingold, C. C. MacDuffee, H. F. MacNeish, Mina S. Rees, W. H. Roever, Lao G. Simons, L. L. Smail, F. M. Weida.

Pi Mu Epsilon of the University of Pennsylvania

The University of Pennsylvania Chapter of Pi Mu Epsilon has enjoyed the year 1931–1932 under the leadership of Dr. G. C. Chambers as director. The officers for the year were: Dr. H. M. Lufkin, Vice Director; Miss M. W. Bell, Secretary; Mr. P. A. Knedler, Treasurer; Dr. H. B. Evans, Chairman of Scholarship Committee; Dr. J. A. Shohat, Chairman of Program Committee.

During the year thirty-two new members were initiated, seventeen in December, and fifteen in April. This makes the total number of active members 252.

The meetings and programs were as follows:

October 16, 1931: "Some aspects of modern Physics" by Dr. E. E. Witmer.
November 20, 1931: "Logarithms" by Mr. B. B. August.
December 15, 1931: "Alignment charts" by Mr. O. Schuck.
January 15, 1932: "Pi and e" by Miss R. E. Cohen.
February 19, 1932: "The place of Mathematics in Physics" by Dr. W. F. G. Swann.
March 18, 1932: "Gyroscopes" by Mr. P. M. Field.
April 15, 1932: "Some interesting curves" by Miss E. K. Clark.
May 6, 1932: "Geometries" by Dr. W. M. Smith.

The February meeting, when Dr. W. F. G. Swann, director of the Bartol Research Foundation of the Franklin Institute, spoke and the May meeting when Dr. W. M. Smith of Lafayette College spoke, were open meetings.

It is the custom of the chapter to award two prizes annually for the best papers presented during the year by undergraduates. At the May meeting, the prizes, each $12.50, were awarded to Miss E. K. Clark and Miss R. E. Cohen.

During the year special problems were offered for solution by any undergraduate student. A prize of $5.00 was offered for the best solution. On February 19th, Mr. B. Miller received the prize for the first problem, and on April 15th, Mr. B. August was announced as winner of the prize for the second problem. The winners of these prizes presented their solutions at meetings of the fraternity.

The events for the year closed with the annual picnic held Saturday, May 21st at Castle Rock.

M. W. BELL, *Secretary*

Pi Mu Epsilon of Pennsylvania State College

Our chapter submits the following outline of events. The officers for the academic year 1931–1932 were: Coleman Herpel, '32, Director; Richard Baker, '32, Vice Director; E. M. Fry, '32, Secretary; Miss Gladys Quigg, Instructor in Mathematics, Treasurer; Dr. C. A. Rupp, Associate Professor in Mathematics, Librarian.

The meetings and programs were as follows:

October 21, 1931: "Signum functions" by Dr. C. C. Wagner.

November 10, 1931: Business meeting. Ten students were elected.

December 16, 1931: Initiation and banquet. About fifty members were present.

January 6, 1932: "Conditional convergence of infinite series" by Dr. T. C. Benton.

February 11, 1932: "Some eccentricities of mathematics" by E. Ross.

March 2, 1932: "The special Relativity Theory" by L. R. Turner; "Empirical equations" by H. L. VanVelzer.

April 14, 1932: "Conics from a projection standpoint" by Dr. T. Cohen.

April 28, 1932: Business meeting. Five students were elected. New officers were elected.

May 9, 1932: "Foundation of Point Set Theory" by Professor R. L. Moore.

May 20, 1932: Cabin party.

E. M. FRY, *Secretary*

LOCAL MATHEMATICS CLUBS

The Mathematics Club of North Carolina College

The officers for the year 1931–1932 were: Edna Livingston, President; Katherine Nowell, Vice President; Blanche Fisher, Secretary-Treasurer; Eleanor Shelton, Chairman of the Program Committee; Miss Emily Watkins, Faculty Advisor.

The meetings and programs were as follows:

October 1931: "1931 as a centennial year in the history of Mathematics." "Birth of Hippocrates" by Jerrie Arthur; "Death of Democritus" by Lollie Boyd; "Publication of *Clavis Mathematics*" by Etta Lowry; "Thomas Harriot" by Audrey James; "Death of Briggs" by Inez Reeves; "Birth of Amedee Mannheim" by Lottie Kennedy; "Birth of James Maxwell" by Olga Frisard; "Cauchy's discovery" by Eddis Byers; "Death of Sophie Germain" by Shirley Hall.

November 1931: "The theory of limits" by Virginia Allen; "The star polygon" by Julia McLendon. The meeting closed with the singing of the club's song.

December 1931: "The solar system" by Lucile Joyner; "Best known planets" by Hazel Goodman. The last talk was illustrated by slides.

January 1932: "Possibilities of a mathematics major in the actuarial departments" by Miss Elizabeth Hall from Jefferson Standard Life Insurance Company; "Clerical work of insurance companies" by Miss Elizabeth Case of the Jefferson Standard Life Insurance Company.

February 1932: Initiation of new members. The program consisted of mathematical puzzles and games.

March 1932: The program consisted of a memorial to one of our first Professors of Mathematics, Miss Gertrude Whittier Mendenhall.

April 1932: "History of Pi" by Lois Siler; "History of e" by Margaret Brown; "The transcendence of Pi and e" by Etta Lowry.

May 1932: The program consisted of a playlet called "Mathematics in Nature" and the cutting of mathematical figures. At this meeting, officers for the academic year 1932–1933 were elected.

ELEANOR SHELTON, *Chairman of Program Committee*

The Mathematics Club of the University of Colorado

The officers for 1931–1932 were: Howard James, President; Doris Huddleston, Vice President; Helen Wirz, Secretary and Treasurer.

The primary purpose of the club is "to point out to everyone the interest and value of vigorous thinking." The club meets once every two weeks and approximately half the meetings are open to all students.

The meetings and programs were as follows:

October 8, 1931: Business meeting.
October 22, 1931: Open meeting. "Mathematical approximations and their applications to the Quantum Theory" by Mr. Louis Strait.
November 15, 1931: Open meeting. "Algebraic differentiation" by Mr. Richard Furr.
January 11, 1932: Open meeting. "The theory of numbers" by Professor Emeritus Ira M. DeLong
January 28, 1932: Closed meeting. "The Geometry resulting from the rejection of the parallel postulate" by Mr. Victor Reno.
February 18, 1932: Open meeting. "The fourth dimension" by Dr. Kempner.
March 3, 1932: Closed meeting. "Famous mathematicians" by Miss Betty Neville.
April 21, 1932: Closed meeting. Business meeting.
May 11, 1932: Closed meeting. Business and election of officers for the academic year 1932–1933.

JOHN R. LACHER, *Secretary*

The Mathematics Club of Rutgers University

The officers of the Rutgers University Mathematics Club for the year 1931–1932 were as follows: Clifton L. Hickok, President; Richard N. Coan, Vice President; Milton Friedman, Secretary-Treasurer. The annual election of officers occurs at the May meeting.

Membership in the Club is open to all students above the Freshman year whose major interest is in Mathematics. This year we had twenty-seven active members.

The following gives the list of the papers read at our seven regular meetings:

October 15, 1931: "Congruences" by R. N. Coan; "Number theory" by H. S. Grant; "Theorem of Appolonius" by J. H. MacDonough.
November 19, 1931: "The earliest mathematics" by E. P. Starke; "Development of demonstrative geometry in Greece" by C. L. Hickok; "Early developments in the Theory of Numbers" by M. Friedman.
December 17, 1931: "Contributions to arithmetic and algebra during the Renaissance" by L. S. Stout; "The development of the Theory of Equations" by W. Weisbrot; "The development of algebraic symbolism" by S. Fenichel; "There is a fifth degree equation which has no solution in radicals" by H. S. Grant.
February 18, 1932: "Descartes and Analytic Geometry" by I. Kaufmann; "Early Projective Geometry" by H. Geller; "The theory of probability" by J. Chernick; "To find directrices and foci without transformation of the general equation of the second degree in two variables" by C. R. Wilson.
March 17, 1932: "The beginnings of Calculus among the Greeks" by C. L. Hickok; "Some famous paradoxes of Zeno" by L. H. Bunyan; "Leibnitz and his calculus" by T. Raiser; "Newton and his calculus" by W. Ward.
April 21, 1932: "Contributions of Euler and Gauss" by R. N. Coan; "Contributions of Laplace and Cauchy" by G. Abbott; "Fractional differentiation" by H. B. Huntley.
May 19, 1932: Generalized algebras" by M. Friedman; "Introduction to Projective Geometry" by C. L. Hickok; "Orthocentric groups of points and counterpoints" by R. Morris.

In conjunction with the mathematics club of the New Jersey College for Women there are held annually two special meetings open to the public. The programs this year were:

December 4, 1931: "A finite geometry" by Professor Alonzo Church of Princeton University.
April 27, 1932: "The early days of the American Mathematical Society" by Professor Thomas Fiske of Columbia University.

MILTON FRIEDMAN, *Secretary*

PROBLEMS AND SOLUTIONS

EDITED BY B. F. FINKEL, OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr. Dept. Box 35, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. Problems may be submitted unaccompanied by their solutions.

PROBLEMS FOR SOLUTION

E 31. *Proposed by W. R. Ransom, Tufts College.*

Point P is on the line AB, one N-th of the way from A to B, and point Q is on the line AC, one N-th of the way from A to C. PC intersects QB at R. Prove that the area of the triangle ABC is $N(N+1)/2$ times the area of the quadrilateral $APRQ$.

E 32. *Proposed by E. C. Kennedy, College of Mines, El Paso, Texas.*

Prove that $\int \sec x\,dx = -2i \tan^{-1} e^{ix} + C$, where $i = \sqrt{-1}$.

E 33. *Proposed by Arthur Haas, Thomas Jefferson High School, Brooklyn, N. Y.*

The points A and B are any two points not in the plane M. Find the locus of the point X, in M, such that the lines AX and BX make equal angles with M.

E 34. *Proposed by V. F. Ivanoff, San Francisco, California.*

Solve $X^{-X} = (-X)^X$ and justify the number of solutions.

E 35. *Proposed by W. B. Campbell, Rangoon, Burma.*

Show that, under appropriate conditions, the limit as n approaches minus one, of the integral $\int_a^b X^n dX$, is equal to the integral $\int_a^b X^{-1} dX$.

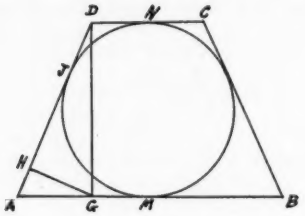
SOLUTIONS

E 11. [1932, 606] *Proposed by W. R. Ransom, Tufts College.*

This problem was offered by Mr. Francis of Exeter at a meeting of the Association of Teachers of Mathematics in New England in 1914.

Circumscribed about a circle is an isosceles trapezoid, $ABCD$, in which $DC<AB$, and $AD=BC$. Two perpendiculars are drawn; DG perpendicular to AB at G, and GH perpendicular to AD at H. Show that DA, DG and DH are the arithmetic, geometric, and harmonic means, respectively, between the pair of parallel sides AB and CD.

Solution by Theodore Lindquist, Michigan State Normal College.



From the diagram, $JD=DN=\frac{1}{2}DC$, and $AJ=AM=\frac{1}{2}AB$; so $DA=\frac{1}{2}(AB+DC)$. Furthermore, $DG^2=DA^2-AG^2=[\frac{1}{2}(AB+DC)]^2-[\frac{1}{2}(AB-DC)]^2=AB\cdot DC$, so that $DG=\sqrt{(AB\cdot DC)}$. Finally, from the similar right triangles, AGD and GHD,

$$DG/HD=DA/DG, \text{ so } 1/HD=DA/DG^2=\tfrac{1}{2}(AB+DC)/(AB\cdot DC)$$
$$=\tfrac{1}{2}[(1/DC)+(1/AB)].$$

Also solved by C. A. Barnhart, W. E. Buker, Mannis Charosh, L. S. Johnston and C. C. Richtmeyer.

E 12. [1932, 606] *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Two coplanar right triangles, AOC and BOC, have the common hypotenuse OC. Using vector methods, express the vector OC in terms of the vectors OA and OB.

Solution by W. R. Ransom, Tufts College.

Represent the vectors OA, OB and OC by the letters A, B and C, respectively. The common perpendicular to A and B is $A\times B$, and the vector C can be expressed as $A+x(A\times B)\times A$, or as $B-y(A\times B)\times B$. Equating these and expanding the triple vector products, we get

$$A+x(BA^2-AA\cdot B)=B+y(AB^2-BB\cdot A).$$

Solving for x, substituting into the first expression for C, and reducing, we get

$$C=[A(A\cdot BB^2-A^2B^2)+B(A\cdot BA^2-A^2B^2)]/[(A\cdot B)^2-A^2B^2].$$

Also solved by Mannis Charosh, T. C. Esty and the proposer.

Advanced Problems

Send all communications about Advanced Problems and Solutions to B. F. Finkel, Springfield, Mo. All manuscript should be typewritten, with double spacing, and with margins at least one inch wide.

Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would

also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3607. *Proposed by J. B. Reynolds, Lehigh University.*

A sphere of radius r and density k times that of water sinks from rest tangent at the top within a fixed sphere of radius R filled with water. Find the velocity of the sinking sphere when its center coincides with that of the fixed sphere. Find the acceleration of the sinking sphere.

3608. *Proposed by N. A. Court, University of Oklahoma.*

The vertex of a positive angle fixed in magnitude describes a given straight line, while the initial line constantly passes through a fixed point. Find the envelope of the second side.

3609. *Proposed by C. C. Carter, Bluffs, Illinois.*

Determine the area of a quadrilateral in terms of its sides, given in the order a, b, c, d, for which the diagonals are equal and at right angles. Is a solution possible if one of the sides, for example b, is not given?

3610. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Determine the position of a cylinder of revolution which cuts the vertical and horizontal planes in equal ellipses similar to a given ellipse.

3611. *Proposed by J. Rosenbaum, Milford, Conn.*

If each of the face angles at a vertex of a tetrahedron is a right angle, the square of the area of the face opposite that vertex is equal to the sum of the squares of the areas of the faces adjacent to that vertex.

Is the converse true?

SOLUTIONS

3516. [1931, 539]. *Proposed by Norman Miller, Queen's University.*

Prove that

$$\lim_{\substack{a_1 \to a \\ a_2 \to a \\ \cdots \\ a_{n+1} \to a}} \frac{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} & a_2^r \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{n-1} & a_{n+1}^r \end{vmatrix}}{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^n \end{vmatrix}} = \frac{r(r-1)\cdots(r-n+1)}{n!} a^{r-n}.$$

Solution by J. M. Feld, Brooklyn College of the City of New York.

Let equation

$$c_0 + c_1x + \cdots + c_nx^n = x^r \quad (r > n+1) \tag{1}$$

have as $n+1$ of its roots $a_i (i=1, \cdots, n+1)$. We can thus obtain the system

$$c_0 + c_1a_i + \cdots + c_na_i^n = a_i^r \quad (i = 1, \cdots, n+1). \tag{2}$$

Solving (2) for c_n we find

$$c_n = f(a_1, a_2, \cdots, a_{n+1}) = \frac{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} & a_1^r \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} & a_2^r \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^{n-1} & a_{n+1}^r \end{vmatrix}}{\begin{vmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^n \\ 1 & a_2 & a_2^2 & \cdots & a_2^n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & a_{n+1} & a_{n+1}^2 & \cdots & a_{n+1}^n \end{vmatrix}}$$

As $a_i \to a$, $(i=1, \cdots, n+1)$, c_n approaches

$$\lim_{a_i \to a} f(a_1, a_2, \cdots, a_{n+1}) \quad (i = 1, \cdots, n+1).$$

Evidently this limit is the coefficient of x^n in (1) if the $n+1$ roots a_i all equal a. If (1) has $n+1$ equal roots, a, the equation obtained after differentiating (1) n times will be satisfied by $x=a$. Thus

$$n!c_n = r(r-1)\cdots(r-n+1)a^{r-n}$$

which proves the theorem

A Note by Otto Dunkel. Since the independent variables a_i are to approach a in any manner, there should be a definition of $c_0, c_1, \cdots, c_n$ for the cases where several or all of the arguments a_i are equal. If we then assume that the c_i's approach finite limits, the above method determines the limit value of c_n.

A proof of a generalization may be given as follows: Let $f(x)$ be a real function of the real variable x which possesses all derivatives up to and including the nth, $f^n(x)$, for an x-interval which contains $a, a_1, a_2, \cdots, a_{n+1}$; and let $f^n(x)$ be continuous in this interval. Denote the determinant in the denominator of the problem by V, and denote by U the determinant obtained by replacing a_i^n in the last column by $f(a_i)$. The generalization consists in replacing x^r of the problem by $f(x)$. Suppose first that no two a_i's are equal. We can then determine uniquely a set of functions of the a_i's, $c_0, c_1, \cdots, c_n$ such that

$$\phi(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n - f(x) \tag{3}$$

vanishes for the $n+1$ distinct values $a_1, a_2, \cdots, a_{n+1}$. The function $\phi(x)$ can be written as a determinant of order $n+2$. Hence the nth derivative of $\phi(x)$ is zero for $x=\theta$, where θ lies between the smallest and largest a_i. We have then

$$c_n = \frac{f^n(\theta)}{n!} = \frac{U(a_1, a_2, \cdots, a_{n+1})}{V(a_1, a_2, \cdots, a_{n+1})}. \tag{4}$$

Suppose that sets of the a_i's are equal, say a_1 occurs m_1 times, a_2 occurs m_2 times, etc., where $a_1, a_2, \cdots, a_t$ are distinct and $m_1+m_2+\cdots+m_t=n+1$. We assign a value to U/V for this case as follows: determine $\bar{c}_0, \bar{c}_1, \cdots, \bar{c}_n$ so that the expression similar to (3) for $\phi(x)$ together with its derivatives up to and including the (m_1-1)th vanishes at $x=a_1$; similarly for $a_2, a_3, \cdots, a_t$. We then find as before

$$c_n = \frac{f^n(\bar{\theta})}{n!} = \frac{\overline{U}(a_1, a_2, \cdots, a_t)}{\overline{V}(a_1, a_2, \cdots, a_t)}, \tag{5}$$

where $\overline{V}$ is obtained from V by replacing in the latter certain of its rows by their derivatives. Also $\overline{V}$ is not zero as long as $a_1, a_2, \cdots, a_t$ are distinct. In the case of several or all a_i's equal we define the value of U/V as that of $\bar{c}_n$ or $\overline{U}/\overline{V}$. With this definition we have in all cases $U/V=f^n(\theta)/n!$, where θ lies between the smallest and largest a_i. Since $f^n(x)$ is continuous and θ approaches a in some manner, we have then

$$\lim_{a_1, a_2, \cdots \to a} \frac{U}{V} = \frac{f^n(a)}{n!}.$$

The above method was given by W. H. Echols in his paper, *On certain determinant forms and their applications*, in Annals of Math., vol. 6, 1892, pp. 105–126. In this paper the function denoted by $\phi(x)$ is written as a determinant of order $n+2$.

Complex Variables. Suppose that $f(z)$ is an analytic function of z within a circle with a as center and radius r, and let $a_1, a_2, \cdots, a_{n+1}$ be any values of z within this circle. Then in the determinant U we may develop each element a_i^m and $f(a_i)$ in series of powers of (a_i-a). Since we have only a finite number of multiplications and additions of absolutely convergent series we obtain an analytic function of $a_1, a_2, \cdots, a_{n+1}$ developed in a multiple power series. The form of the development may be readily seen by developing first only the last column of elements, $f(a_i)$. We obtain an infinite series

$$\sum A_j f^j(a)/j!,$$

where A_j is obtained by replacing in V the elements a_i^m of the last column by $(a_i-a)^j$. It is then clear that $A_j=0$ for $j<n$, and that $A_n=V$. From the form of V we have at once $V(a_1, a_2, \cdots, a_{n+1})=V(a_1-a, a_2-a, \cdots, a_{n+1}-a)$. We now develop the remaining elements of A_j and after certain simple reductions we find that A_j has the same first n columns as $V(a_1-a, a_2-a, \cdots, a_{n+1}-a)$,

but the $(n+1)$th column of A_j contains jth powers instead of nth powers. Thus A_j is divisible by V and the quotient is a homogeneous polynomial in the differences $a_1-a, a_2-a, \cdots$, of degree $j-n$. Hence

$$U = V\left[\frac{f^n(a)}{n!} + E\right],$$

where E is an analytic function of the a_i's. Hence the brackets give a definition of U/V for all cases. Since E approaches zero as the a_i's approach a in any manner, we obtain the same result as before for the limit of U/V.

In the given problem $f(x)=x^r$, and if r is an integer, positive, negative or zero, the problem is simplified. If r is a positive integer, U is divisible by V if no two of the a_i's are equal; and we define U/V as this polynomial quotient. The existence of the limit is then apparent, and it may be found as above or otherwise. If r is a negative integer, we exclude the case of $a=0$. We may write $U=(a_1a_2\cdots a_{n+1})^rW$, where W is a determinant with a_i^{-r}, a_i^{-r+1}, $\cdots$, a_i^{-r+n-1}, 1 in the ith row. Again W is divisible by V if no two a_i's are equal, and we again obtain a definition of U/V as $(a_1a_2\cdots a_{n+1})^rW/V$, where the factor W/V is a polynomial. The existence of a limit is again apparent.

If r is not an integer and a is not zero, x^r is analytic within a suitable circle with a as center which does not contain $x=0$, after selecting a determination of x^r for this circle.

Also solved by A. S. Householder, who gave the reference to Echols's work cited above.

3552 [1932, 300]. *Proposed by J. M. Feld, Brooklyn College of the City of New York.*

Prove that

$$\begin{vmatrix} e_{11} & 0 & 0 & \cdots 0 & A_1 \\ e_{21} & e_{22} & 0 & \cdots 0 & A_2 \\ e_{31} & e_{32} & e_{33} & \cdots 0 & A_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ e_{n1} & e_{n2} & e_{n3} & \cdots e_{n,n-1} & A_n \end{vmatrix}$$

$$= A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s},$$

where (1) $e_{ij}=1$ if j is a divisor of i and $e_{ij}=0$ if j is not a divisor of i; (2) p_i, p_j, $\cdots$, are the distinct prime factors of n.

Solution by N. H. McCoy, Smith College.

In the given determinant, subtract from the n-th row each of the rows n/p_i, add each of the rows n/p_ip_j and so on, and denote the resulting determinant by D. Thus D is equal to the original determinant.

Let a_{nj}, $(j=1, 2, \cdots, n)$ denote the elements of the n-th row of D. Then if we let $e_{ij}=0(i<j)$, we may write

$$a_{nk} = e_{nk} - \sum e_{n/p_i,k} + \sum e_{n/p_ip_j,k} - \cdots + (-1)^s e_{n/p_1p_2\cdots p_s,k}, \quad (k=1, 2, \cdots, n-1),$$

$$a_{nn} = A_n - \sum A_{n/p_i} + \sum A_{n/p_ip_j} - \cdots + (-1)^s A_{n/p_1p_2\cdots p_s}.$$

If k is not a divisor of n, clearly $a_{nk}=0$. Suppose that k divides n. Let $n=p_1^{t_1}p_2^{t_2}\cdots p_s^{t_s}$, $k=p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$, and suppose that $r_i<t_i$ for just m values of i. Then

$$a_{nk} = 1 - m + \binom{m}{2} - \cdots + (-1)^m = (1-1)^m = 0, \quad (k=1, 2, \cdots, n-1).$$

Hence a_{nn} is the only non-vanishing element in the last row of D, and since $e_{ii}=1$, we have $D=a_{nn}$.

3556 [1932, 301]. *Proposed by the late Artemas Martin, Washington, D. C.*

A hollow sphere, external and internal radii R and r, rolls down an inclined plane in time t; after the cavity is half filled with water it rolls down the same plane in time t'. Determine the specific gravity of the sphere.

See the *Annals of Mathematics*, vol. 8 (1894), p. 104.

Solution by William Hoover, Columbus, Ohio.

Let M and k be the mass and radius of gyration of the hollow sphere; m, the mass of the water, l the length of the inclined plane; α, the inclination of the plane; x, the distance the center of the sphere has moved in the time t from the beginning of the motion; θ, the angle through which the sphere has rotated in this time. Assuming that there is no friction between the water and the inner surface of the sphere, we have in the second case the energy equation

$$\tfrac{1}{2}M(\dot{x}^2 + k^2\dot{\theta}^2) + \tfrac{1}{2}m\dot{x}^2 = g\sin\alpha(M+m)x, \tag{1}$$

or, since $\dot{x}=R\dot{\theta}$,

$$\begin{aligned}[M(R^2+k^2)+mR^2]\dot{x}^2 &= 2g\sin\alpha R^2(M+m)x,\\ [M(R^2+k^2)+mR^2]\ddot{x} &= g\sin\alpha R^2(M+m),\end{aligned} \tag{2}$$

where the second equation results from the derivative of the first. After integration we get

$$2[M(R^2+k^2)+mR^2]l = g\sin\alpha R^2(M+m)t'^2, \tag{3}$$

observing that for $t=0$, $x=0$, $\dot{x}=0$, and that for $t=t'$, $x=l$.

For the first case of motion, we replace in (3) t' by t and set $m=0$. Hence after dividing the two results and thus eliminating l, g and $\sin\alpha$, we obtain

$$M\left[1-\left(\frac{t'}{t}\right)^2\right] = m\left[\left(\frac{t'}{t}\right)^2 - \frac{R^2}{R^2+k^2}\right]. \tag{4}$$

We also have

$$k^2 = \frac{2}{5}\frac{R^5 - r^5}{R^3 - r^3}, \quad M = \frac{4}{3}\pi\delta'(R^3 - r^3), \quad m = \frac{2}{3}\pi\delta r^3,$$

where δ' and δ are the densities of M and m. These substitutions in (4) give the desired equation in δ'/δ.

3557 [1932, 359]. *Proposed by R. E. Moritz, University of Washington.*

Given a free flow of water from a vertical circular aperture, flowing partially full. The radius of the aperture is r, the depth of the water above the center of the aperture is a, the coefficient of entry is k. Required the rate of discharge under the action of gravity.

Solution by the Proposer.

Let dA represent the area of a cross-section of a differential element of flow, h its hydraulic head, then its velocity of discharge due to gravity is $v=\sqrt{2gh}$, where $g=32.2$, and the differential element of discharge due to the element of flow is $dQ=k\cdot v\cdot dA$.

If now (x, y) are the Cartesian coordinates of a point on the circumference of the aperture, referred to the center of the aperture as origin, and the horizontal direction for the direction of the x-axis, then

$$x^2 + y^2 = r^2, \quad dA = 2xdy, \quad v = \sqrt{2g(a - y)};$$

hence

$$dQ = k\sqrt{2g(a - y)}\cdot 2xdy = 2k\sqrt{2g}\cdot\sqrt{(a - y)(r^2 - y^2)}\cdot dy$$

and

$$Q = 2k\sqrt{2g}\int_{-r}^{a}\sqrt{(a - y)(r^2 - y^2)}\cdot dy. \tag{1}$$

To evaluate this integral, let $x=\mathrm{sn}^{-1}[(y+r)^{1/2}(a+r)^{-1/2}, k]$, where $k^2=(a+r)/2r$, then when $y=-r$, $x=\mathrm{sn}^{-1}(0, k)=0$, and when $y=a$, $x=\mathrm{sn}^{-1}(1, k)=K$; and

$$a - y = (a + r)\,\mathrm{cn}^2 x, \quad r^2 - y^2 = 2r(a + r)\,\mathrm{sn}^2 x\,\mathrm{dn}^2 x,$$
$$dy = 2(a + r)\,\mathrm{sn}\,x\,\mathrm{cn}\,x\,\mathrm{dn}\,x\,dx$$

$$\int_{-r}^{a}\sqrt{(a - y)(r^2 - y^2)}dy = 2(a + r)^2\sqrt{2r}\int_0^K \mathrm{cn}^2 x\,\mathrm{sn}^2 x\,\mathrm{dn}^2 x\,dx, \tag{2}$$

$$= 2(a + r)^2\sqrt{2r}\int_0^K [\mathrm{sn}^2 x - (1 + k^2)\,\mathrm{sn}^4 x + k^2\,\mathrm{sn}^6 x]dx,$$

$$= \frac{2\sqrt{2r}}{15}[(a - 3r)(r - a)K + 2(a^2 + 3r^2)E],$$

by the use of reduction formulas such as Peirce's formula 567. K and E are the complete Elliptic Integrals of the first and second class respectively. When (2) is substituted in (1) the final result is obtained.

3558 [1932, 359]. *Proposed by W. H. Rasche, Virginia Polytechnic Institute.*

Of a plane triangle given one side, the length of the bisector of the opposite angle, and the altitude on the given side; state and prove the ruler-compass construction of the triangle.

Solution by Banesh Hoffmann, University of Rochester.

Construction. Set up the right triangle AKM having angle $AKM = 90°$, AK = length of altitude and AM = length of bisector. Draw AN perpendicular to AM to meet MK in N and find the mid-point, O, of MN. Draw MR of length a ($2a$ = length of base of required triangle) perpendicular to MK and mark off Q on MK such that $OQ = OR$ (Q nearer M than N). Make $QC = QB = a$ on MK; then ABC is the required triangle.

Proof. We must now show that angle BAC is actually bisected by AM. From the indicated construction, we have

$$QB^2 = QC^2 = RM^2 = RO^2 - OM^2 = OQ^2 - OM^2 = (OQ - OM)(OQ + OM)$$
$$= QM \cdot QN.$$

Therefore B, M, C, N is an harmonic set of points, and, since MAN is a right angle, AM and AN are the internal and external bisectors of angle BAC.

Note by the Editors. In the above construction it is assumed that the given bisector is the interior bisector and that it is greater than the given altitude. The solution is then unique. If the given bisector is the exterior bisector, Q should be taken nearer N than M. If the given altitude and bisector are equal, the above construction breaks down; but it can be replaced by an obvious and trivial one in this special case.

Also solved by Gertrude Blanch, W. R. Church, Roy MacKay, J. S. Miller, R. K. Morley, William Orange, A. Pelletier, H. D. Ruderman, and Paul Wernicke.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending items to Professor J. H. Weaver, Ohio State University, Columbus, Ohio.

The faculty of the school of mathematics of the Institute for Advanced Study and the faculty of the department of mathematics at Princeton will issue the Annals of Mathematics under a board of editors selected from each of the two institutions.

Professor Herman Weyl, professor of mathematics at the University of Göttingen, and Dr. J. W. Alexander, professor of mathematics at Princeton, have been appointed professors in the Institute for Advanced Study.

Dr. Solomon Lefschetz, professor of mathematics at Princeton, has been appointed to the Henry Burchard Fine chair of mathematics to succeed Professor Oswald Veblen.

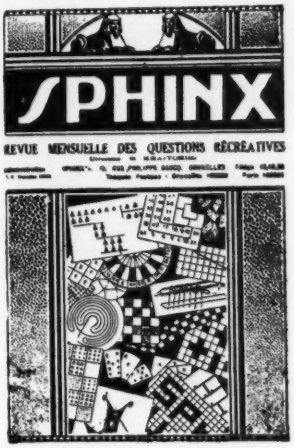
Mr. H. P. Fawcett has been appointed assistant professor of mathematics in the department of the University Laboratory High School at the Ohio State University.

Mr. F. C. Hall has been appointed an instructor at New York University.

Professor R. L. Börger, head of the department of mathematics at Ohio University, died December 26, 1932. He was a charter member of the Association.

Harlan Wilbur Fiske, of Kensington, Maryland, magnetician and chief of the magnetic land survey, department of terrestrial magnetism, Carnegie Institution at Washington, died December 26, 1932.

Ormond Stone, professor emeritus of the University of Virginia, was struck and instantly killed by an automobile January 17, 1933. With his passing the University of Virginia loses its oldest member, and astronomy a notable figure. He began his career as an astronomer at the Naval Observatory. From there he went to the Observatory at Cincinnati, and later he was called to the University of Virginia. Although primarily an astronomer, Professor Stone was very much interested in mathematics. He showed concrete evidence of this interest by founding the *Annals of Mathematics* and acting as its first editor. He was a charter member of the Mathematical Association.



SPHINX

Revue Mensuelle des Questions Récréatives

Directeur: M. Kraïtchik
Laureat de l'Institut de France

Revue unique dans son genre dans le monde entier

Abonnement—$1.00

Administration: Bruxelles (Belgium), 75 Rue Philippe-Baucq

Publishers: G. E. STECHERT & CO., New York—DAVID NUTT, London—NICOLA ZANICHELLI, Bologna—FÉLIX ALCAN, Paris—AKADEMISCHE VERLAGSGESELLSCHAFT, m. b. H. Leipzig—RUIZ HERMANOS, Madrid—F. MACHADO & CIA., Porto—THE MARUZEN COMPANY, Tokyo

1933 27th Year

"SCIENTIA"

INTERNATIONAL REVIEW OF SCIENTIFIC SYNTHESIS
Published every month (each number containing 100 to 120 pages)
Editors: **F. BOTTAZZI - G. BRUNI - F. ENRIQUES**
General Secretary: **Paolo Bonetti.**

IS THE ONLY REVIEW the contributors to which are really international

IS THE ONLY REVIEW that has a really world-wide circulation.

IS THE ONLY REVIEW of scientific synthesis and unification that deals with the fundamental questions of all sciences: mathematics, astronomy, geology, physics, chemistry, biology, psychology, ethnology, linguistics; history of science; philosophy of science.

IS THE ONLY REVIEW that by means of enquiries among the most eminent scientists and authors of all countries (*On the philosophical principles of the various sciences; On the most fundamental astronomical and physical questions of current interest; On the contribution that the different countries have given to the development of various branches of knowledge; On the more important biological questions,* etc. etc.), studies all the main problems discussed in intellectual circles all over the world, and represents at the same time the first attempt at an international organization of philosophical and scientific progress.

IS THE ONLY REVIEW that among its contributors can boast of the most illustrious men of science in the whole world.

The articles are published in the language of their authors, and every number has a supplement containing the French translation of all the articles that are not French. The review is thus completely accessible to those who know only French. *(Write for a free copy to the* General Secretary of "Scientia," Milan, *sending 12 cents in stamps of your country, merely to cover packing and postage).*

SUBSCRIPTION: $10.00 Post free

Substantial reductions are granted to those who take up more than one year's subscription.

For information apply to "SCIENTIA" Via A. De Togni, 12 - Milano 116 (Italy)

CONTENTS

The December Meeting of the Maryland-District of Columbia-Virginia Section. By F. M. WEIDA 189

Eliakim Hastings Moore—An Appreciation. By H. E. SLAUGHT 191

Curves Arising from a Single Infinity of Triangles. By MABEL M. YOUNG 196

Thermostatic Control. By J. L. SYNGE 202

A Convergence Test and a Remainder Theorem. By W. F. LIBBY 216

Power Sums of Roots. By H. D. GROSSMANN 219

QUESTIONS, DISCUSSIONS, AND NOTES: Recursion Relations in Certain Expansion Coefficients, by R. S. UNDERWOOD; The Cubic $x^2+y^2+z^2-2xyz=1$, by A. S. MERRILL; On a Certain Transformation of Infinite Series, by J. A. SHOHAT 222

RECENT PUBLICATIONS: New Books Received; Reviews by H. E. SLAUGHT, R. L. WILDER, R. D. CARMICHAEL, F. R. MOULTON, SAMUEL BOROFSKY, B. F. FINKEL, B. P. GILL 229

MATHEMATICS CLUBS: Club Activities 238

PROBLEMS AND SOLUTIONS: Elementary Problems for Solution, E31–E35; Solutions, E11, E12; Advanced Problems for Solution, 3607–3611; Solutions, 3516, 3552, 3556–3558 241

NEWS AND NOTICES 250

DIRECTORY

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, W. B. CARVER, White Hall, Cornell University, Ithaca, N.Y.

BOOKS FOR REVIEW should be addressed to R. A. JOHNSON, Brooklyn College, 66 Court Street, Brooklyn, N.Y.

BUSINESS CORRESPONDENCE should be addressed to the SECRETARY-TREASURER of the Association, W. D. CAIRNS, 33 Peters Hall, Oberlin, Ohio.

CHANGE OF ADDRESS: Members should send notice of any change of address to the SECRETARY-TREASURER, W. D. CAIRNS, Oberlin, Ohio, before the 10th of each month.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Seventeenth Summer Meeting of the Association, Chicago, Ill., June 20-22, 1933.

The following is a list of the sections of the Association, with dates of those section meetings which have been scheduled for 1933 and reported to the Secretary.

ILLINOIS.
INDIANA, Bloomington, May 5-6.
IOWA, Cedar Rapids, Apr. 21-22.
KANSAS, Topeka, Feb. 11.
KENTUCKY, May.
LOUISIANA-MISSISSIPPI, Ruston, La., Mar. 3-4.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Charlottesville, Va., May 13.
MICHIGAN, Ann Arbor, Mar. 18.
MINNESOTA.
MISSOURI.
NEBRASKA, Lincoln, Apr. 28.
OHIO, Columbus, Apr. 6.
PHILADELPHIA, Philadelphia, Dec. 2.
ROCKY MOUNTAIN, Fort Collins, Colo., Apr. 14-15.
SOUTHEASTERN, Athens, Ga., March.
SOUTHERN CALIFORNIA, Claremont, Mar. 4.
TEXAS, Dallas, Feb. 11.
WISCONSIN, Beloit, Apr. 8.

AFFILIATED ORGANIZATIONS: THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS. THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS.

PASCAL'S HEXAGRAM WITH CERTAIN ELEMENTS IN MOTION

By W. H. BUNCH, University of Idaho

Notation. We shall refer to the 45 Pascal points as the P-points and the 60 Pascal lines as the p-lines. In all other cases capital letters will represent points and the corresponding lower case letters will represent their polar lines. Numbers will be used as subscripts and when no confusion results they will be used alone. Thus 12 may be the line joining T_1 and T_2, (12,45) may be a point determined by two lines, and (12,45)–(34,56) a line determined by two points. When the numbers are subscripts for the lower case letters, the above notations will have the dual meanings. We shall refer to the vertices of the Pascal hexagon as the T points and tangents to the conic at these points as the t-lines.

Let any one of the six T-points move along the conic and the other five remain fixed; then the following theorems may be established.

THEOREM I. *Fifteen of the 45 P-points will remain unchanged and 30 will move along straight lines determined by the fixed T-points.*

PROOF: Fifteen P-points are determined by the five fixed T-points; hence, they do not move. Six P-points lie on each line determined by any two T-points and the P-points are the intersections of all such lines. Five of these lines pass through each T-point, hence 30 P-points move along straight lines determined by the fixed T-points.

THEOREM II. *Every p-line passes through one and only one of the fifteen fixed P-points, and the other two P-points on each p-line move along two intersecting straight lines and describe perspective point rows upon them.*

PROOF: The three P-points on every p-line are formed by the intersection of the opposite sides of the hexagon, hence the two adjacent sides which pass through any one of the T-points must pass through two different P-points in each p-line. Therefore, two points on each line must move. The remainder of the theorem follows immediately.

THEOREM III. *The intersection of every two p-lines which do not intersect in a P-point describes a conic, which intersects the original conic in one of the fixed T-points and passes through two fixed P-points.*

PROOF: Let point 1 be any of the six T-points and 2, 3, 4, 5, and 6 be the other five. Let any two of these, say 5 and 6, be centers of projective pencils which generate the conic. Then the lines 15 and 16 are corresponding rays. Now let point 1 move along the conic and the lines 15 and 16 will describe perspective[1] point rows upon any two lines which intersect upon the conic excluding the line 56 which is a common ray of the two pencils. Let us choose, for ex-

[1] Lehmer, *Synthetic Geometry*, page 33.